McGill Fall 2024 - MATH580 - Advanced PDEs I
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Contents

1 Introduction to PDEs

Definition 1.1. (General Form of a PDE) The general form of a PDE is given by:

$$F(D^k u, D^{k-1}u, \ldots, Du, u, x) = 0 \in U \tag{1}$$

such that $x \in U$ open subset of $\mathbb{R}^n$, $k, n \in \mathbb{N}$, $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$ is given, and $u : U \to \mathbb{R}$ is the unknown function.

Definition 1.2. (u is a Solution) We say that u is a solution (or classical solution or strong solution) of (1) if all derivatives in (1) exist and are continuous and (1) is satisfied at every point $x \in U$.

Remark 1.1. For time evolution PDEs, we consider $U = \mathbb{R}^n \times I \to \mathbb{R}$ where U is an open subset of $\mathbb{R}^n$ and I is an open interval in $\mathbb{R}$.

Example 1.1.

1. Heat Equation: $u_t - \Delta u = 0$ in $U = \mathbb{R}^n \times I$.
2. Wave Equation: $u_{tt} - \Delta u = 0$ in $U = \mathbb{R}^n \times I$.
3. Schrodinger Equation: $iu_t + \Delta u = 0$ in $U = \mathbb{R}^n \times I$.
4. Navier-Stokes Equation: $\rho(u_t + u \cdot \nabla u) = -\nabla p + \mu \Delta u + f$ in $U = \mathbb{R}^n \times I$.
5. Laplace Equation: $\Delta u = 0$ in $U = \mathbb{R}^n$.
6. Poisson Equation: $-\Delta u = f$ in $U = \mathbb{R}^n$.
7. Transport Equation: $u_t + \nabla \cdot (bu) = 0$ in $U = \mathbb{R}^n \times I$.

Definition 1.3. (PDE Classifications)

1. The order of a PDE is the highest order of the derivatives in the PDE.
2. Linear PDEs are of the form $\sum_{|\alpha| \subseteq k} a_\alpha(x) D^{(\alpha)} u = f(x)$, where $a_\alpha, f : U \to \mathbb{R}$ are given and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index of length $|\alpha| = \alpha_1 + \ldots + \alpha_n$
   (i) That is: $D^{(2,0,1)}u = \partial_{x_1}^2 \partial_{x_3} u, |(2,0,1)| = 3$
3. Semi-linear PDEs are of the form $\sum_{|\alpha|=k} a_\alpha(x) D^{(\alpha)} u = f(D^{k-1}u, \ldots, Du, u, x)$.
4. Quasi-linear PDEs are of the form $\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \ldots, Du, u, x) D^{(\alpha)} u = f(D^{k-1}u, \ldots, Du, u, x)$.
5. Fully non-linear PDEs are any PDE that is not quasi-linear, semi-linear, or linear.

Definition 1.4. (Boundary Value Problems (BVP))

1. Dirichlet BVP: $u = g$ on ∂U.
2. Neumann BVP: $\partial_\nu u = g$ on ∂U.
3. Robin BVP: $\alpha u + \beta \partial_\nu u = g$ on ∂U.

Definition 1.5. (Initial Value Problems (IVP))

1. Cauchy Problem: $u(x, 0) = g(x)$.
2. Initial Boundary Value Problem: $u(x, 0) = g(x)$ and $u = g$ on ∂U.

Definition 1.6. (Well-posed) We say that an IVP is well-posed if there exists a unique solutions and this solution depends continuously on the initial data.

2 First Order PDEs

2.1 The Method of Characteristics

Definition 2.1. (Linear Method of Characteristics)
Set up: a first order linear PDE defined as

$$F(Du, u, x) = 0 \in U \subset \mathbb{R}^n \quad (E)$$

such that $F : (p, z, x) \to F(p, z, x)$, $F \in C^2(\mathbb{R}^n \times \mathbb{R} \times U)$, $u \in C^2(U)$
We need some regularity to apply the method.
Assume $D_p f \neq 0$, let $x : J \subset \mathbb{R} \to U$ be a C^1 curve such that $x'(s) = D_p F(Du(x(s)), u(x(s)), x(s)) \quad (\star_1)$
We define $z(s) = u(x(s))$ and $p(s) = Du(x(s))$
By differentiating $z(s)$ and $p(s)$, with respect to s, we get:

$$z'(s) =< Du(x(s), x'(s)) > \quad (\star_2)$$

and

$$p'(s) = D^2 u(x(s)) x'(s) \quad (\star_3)$$

By differentiating (E) with respect to x we get:

$$D_p F(Du(x), u(x), x) D^2 u(x) + \partial_2 F(Du(x), u(x), x) + D_x F(Du(x), u(x), x) = 0$$

At $x(s)$, by using $(\star_1), (\star_2), (\star_3)$ we have:

$$x'(s) D^2 u(x(s)) + \partial_2 F(p(s), z(s), x(s)) p(s) + D_x F(p(s), z(s), x(s)) = 0$$

We now obtain the system of characteristics:

$$\begin{cases} x'(s) = D_p F(p(s), z(s), x(s)) \\ z'(s) =< p(s), D_p F(p(s), z(s), x(s)) > \\ p'(s) = -\partial_2 F(p(s), z(s), x(s)) - D_x F(p(s), z(s), x(s)) \end{cases} \tag{2}$$

Remark 2.1. In order to have existence and uniqueness, we require that it is Lipschitz continuous, and we get well-posedness with initial conditions.

Remark 2.2. When $F \in C^2(U)$ standard ODE results give local well-posedness with respect to any initial data.

Definition 2.2. (Quasilinear Method of Characteristics) Set up: $< b(u, x), Du >= f(u, x) \in U$ for some $b, f \in C^1(\mathbb{R} \times U)$
$F(p, z, x) =< b(z, x), p > -f(z, x)$
$D_p F(p, z, x) = b(z, x)$ hence the first two equations give:

$$\begin{cases} x'(s) = b(z(s), x(s)) \\ z'(s) =< p(s), b(z(s), x(s)) >= f(z(s), x(s)) \end{cases} \tag{3}$$

In this case, we don't need the third ODE equation. We can also assume that $u \in C^1(U)$ instead of $C^2(U)$.

Example 2.1. $< x, Du >= 0$ in $\mathbb{R}^n \backslash \{0\}$

$$\begin{cases} x'(s) = x \\ z'(s) =< p, x >= 0 \end{cases} \implies \begin{cases} x(s) = x_0 e^s \\ z(s) = z_0 \end{cases}$$

Hence $u(x_0 e^s) = z_0$.
That is, u is constant along $s \mapsto x_0 e^s \quad \forall x_0 \in \mathbb{R}^n \backslash \{0\}$

Definition 2.3. (Linear Time-Evolution Method of Characteristics)
Set up: $F(Du, \partial_t u, u, x, t) = 0 \in U \times I$
In this case, replace $x(s)$ by $(x(s), t(s))$ and define $z(s) = u(x(s), t(s))$, $p_x(s) = Du(x(s), t(s))$, and $p_t(s) = \partial_t u(x(s), t(s))$
Then the characterstic system is:

$$\begin{cases} x'(s) = D_{p_x} F(p_x(s), p_t(s), z(s), x(s), t(s)) \\ t'(s) = \partial_{p_t} F(p_x(s), p_t(s), z(s), x(s), t(s)) \\ z'(s) =< D_{p_x} F(p_x(s), p_t(s), z(s), x(s), t(s)), p_x(s) > + \partial_{p_t} F(p_x(s), p_t(s), z(s), x(s), t(s)) p_t(s) \\ p_x'(s) = -\partial_z F(p_x(s), p_t(s), z(s), x(s), t(s)) p_x(s) - D_x F(p_x(s), p_t(s), z(s), x(s), t(s)) \\ p_t'(s) = -\partial_z F(p_x(s), p_t(s), z(s), x(s), t(s)) p_t(s) - \partial_t F(p_x(s), p_t(s), z(s), x(s), t(s)) \end{cases} \tag{4}$$

Definition 2.4. (Quasilinear Time-Evolution Method of Characteristics)
Set up: $a(u, x, t)\partial_t u + < b(u, x, t), Du >= f(u, x, t) \in U \times I$
We have:

$$\begin{cases} x'(s) = b(z(s), x(s), t(s)) \\ t'(s) = a(z(s), x(s), t(s)) \\ z'(s) = f(z(s), x(s), t(s)) \end{cases} \tag{5}$$

Example 2.2. $\partial_t u + < b(x, t), Du >= 0 \in U \times I$

$$\begin{cases} x' = b(x, t) \\ t' = 1 \\ z' = 0 \end{cases} \implies \begin{cases} x = b(x(s), s + t_0) \\ t = t_0 + s \\ z = z_0 \end{cases}$$

There are two cases we can consider:

1. Case 1: b is a constant
$$x(s) = bs + x_0, \text{ hence } u(bs + x_0, t_0) = z_0$$
That is, u is constant along $s \mapsto (bs + x_0, s + t_0) \quad \forall (x_0, t_0)$

2. Case 2: $b(x, t) = x$
$$x(s) = x_0 e^s, \text{ hence } u(x_0 e^s, s + t_0) = z_0$$
That is, u is constant along $s \mapsto (x_0 e^s, s + t_0) \quad \forall (x_0, t_0)$ which is exponential.

Example 2.3. (Fully Nonlinear; Hamilton-Jacobi)

$$\partial_t u + \frac{1}{2}|Du|^2 = f(x) \in \mathbb{R}^n \times (0, \infty)$$

We have: $F(p_x, p_t, z, x, t) = p_t + \frac{1}{2}|p_x|^2 - f(x)$

$$\begin{cases} x' = p_x \\ t' = 1 \\ z' = p_t + < p_x, p_x >= p_t + |p_x|^2 = f(x) + \frac{1}{2}|p_x|^2 \\ p_x' = Df(x) \\ p_t' = 0 \end{cases}$$

2.2 Boundary Conditions

Set up:

$$\begin{cases} F(Du, u, x) = 0, \text{ in } U \\ u = g, \text{ on } \Gamma \subset \partial U \end{cases} \tag{6}$$

2.2.1 Case of a Flat Boundary

Set up: $\Gamma \subset \mathbb{R}^{n-1} \times 0$, $g \in C^2$, $F \in C^2(\mathbb{R}^n \times \mathbb{R} \times (U \cup \Gamma))$, $u \in C^2(U \times \Gamma)$
In the classical sense that all partial derivatives of $F(p, z, \cdot)$ and u exist, and are continuous at all points in $U \cup \Gamma$.
Let (p, z, x) be a solution of the characteristics equations such that x crosses Γ at $s = 0$.
We define: $x_0 = x(0), z_0 = z(0) = g(x(0)), p_0 = (p_{0,1}, \dots, p_{0,n}) = p(0)$
Differentiating $u = g$ in tangent directions to Γ:

$$p_{0,i} = \partial x_i g(x_0) \quad \forall i \in \{1, \dots, n-1\}$$

Moreover, (p_0, x_0, z_0) satisfy the PDE $F(p_0, x_0, z_0) = 0$
Combining these facts, we obtain the compatibility conditions:

$$\begin{cases} z_0 = g(x_0) \\ p_{0,i} = \partial x_i g(x_0) \quad \forall i \in \{1, \dots, n-1\} \\ F(p_0, z_0, x_0) = 0 \end{cases} \tag{7}$$

We need the derivative with respect to p_n be non-zero as we don't want the characteristics to be tangential to the boundary. To find $p_{0,n}$ in terms of $p_{0,1}, \dots, p_{0,n-1}, z_0, x_0$ it usually suffices to use the PDE $F(p_0, z_0, x_0) = 0$.

Definition 2.5. (Non-characteristic condition) This is guaranteed by the implicit function theorem provided that:

$$\partial p_n F(p_0, z_0, x_0) \neq 0, \quad s \mapsto x(s)$$

2.2.2 Case of time-evolution equations

$$\begin{cases} F(Du, \partial_t u, u, x, t) = 0, \quad \text{in } U \times (0, T) \\ u(\cdot, 0) = u_0, \quad u_0 \in C^2(U) \end{cases}$$

In this case, we define $x(0) = x_0, z(0) = z_0, p_x(0) = p_{0,k}, p_t(0) = p_{0,t}$

Definition 2.6. (Compatibility conditions)

$$\begin{cases} z_0 = u_0(x_0) \\ p_{0,x} = Du_0(x_0) \\ F(P_{0,x}, p_{0,t}, z_0, x_0, t_0 = 0) = 0 \end{cases}$$

Definition 2.7. (Non-characteristic condition)

$$\partial_{p_t} F(p_{0,x}, p_{0,t}, z_0, x_0, 0) \neq 0$$

Example 2.4. (Transport Equation)

$$\begin{cases} \partial_t u + < b(x, t), Du >= 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \\ u(\cdot, 0) = u_0 \quad \text{on } \mathbb{R}^n \end{cases}$$

We have: $F(p_x, p_t, z, x, t) = p_t + < b(x,t), p_x >$

Compatibility condition:

$$\begin{cases} z_0 = u_0(x_0) \\ p_{0,x} = Du_0(x_0) \\ p_{0,t} = - < b(x,0), p_{x,0} > = - < b(x_0,0), Du_0(x_0) > \end{cases}$$

Non-characteristic condition:

$$\partial_{p_t} F(p_{0,x}, p_{0,t}, z_0, x_0, 0) = 1 \neq 0$$

Case 1: b is constant In this case we obtain $u(bs + x_0, s + t_0) = z_0$. Here $t_0 = 0$. By using the compatibility conditions, we then obtain $u(bs + x_0, s) = u_0(x_0)$ and so:

$$\begin{cases} x = bs + x_0 \\ t = s \end{cases} \implies \begin{cases} x_0 = x - bt \\ s = t \end{cases}$$

Hence, $u(x,t) = u_0(x - bt)$

Case 2: $b(x,t) = x$

In this case we obtain $u(x_0 e^s, s + t_0) = z_0$. Again $t_0 = 0$. By using the compatibility conditions, we then obtain $u(x_0 e^s, s) = u_0(x_0)$ and so:

$$\begin{cases} x = x_0 e^s \\ t = s \end{cases} \implies \begin{cases} x_0 = x e^{-s} \\ s = t \end{cases}$$

Hence $u(x,t) = u_0(x e^{-t})$

Example 2.5. (Hamilton-Jacobi)

$$\begin{cases} \partial_t u + \frac{1}{2}|Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(\cdot, 0) = < V_0, \cdot > & \text{on } \mathbb{R}^n \end{cases}$$

We have: $F(p_x, p_t, z, x, t) = p_t + \frac{1}{2}|p_x|^2$

Compatibility conditions:

$$\begin{cases} z_0 = < V_0, x_0 > \\ p_{0,x} = V_0 \\ p_{0,t} = -\frac{1}{2}|p_{0,x}|^2 = -\frac{1}{2}|V_0|^2 \end{cases}$$

Non-characteristic condition:

$$\partial_{p_t} F(p_{0,x}, p_{0,t}, z_0, x_0, 0) = 1 \neq 0$$

Characteristic equations:

$$\begin{cases} x'(s) = p_x(s) \\ t'(s) = 1 \\ z'(s) = \dfrac{1}{2}|p_x(s)|^2 \\ p_x'(s) = 0 \\ p_t'(s) = 0 \end{cases} \implies \begin{cases} x(s) = V_0 s + x_0 \\ t(s) = s \\ z(s) = \dfrac{1}{2}|V_0|^2 s + z_0 = \dfrac{1}{2}|V_0|^2 s + < V_0, x_0 > \\ p_x(s) = p_{x,0} = V_0 \\ p_t(s) = p_{t,0} = -\dfrac{1}{2}|V_0|^2 \end{cases}$$

Hence we have that $u(V_0, s + x_0, s) = \frac{1}{2}|V_0|^2 + < V_0, x_0 >$ And so:

$$\begin{cases} x = V_0 s + x_0 \\ s = t \end{cases} \implies \begin{cases} x_0 = x - V_0 t \\ t = s \end{cases}$$

Which gives $u(x,t) = \frac{1}{2}|V_0|^2 t + < V_0, x - V_0 t >$

2.2.3 Case of a non-flat boundary

Definition 2.8. (C^k) For each $k \in \mathbb{N}$, we say that an open subset Γ of ∂U is C^k if for each $x_0 \in \Gamma, \exists r > 0$ and a C^k-diffeomorphism by $\phi_{x_0} : B(x_0, r) \to \phi_{x_0}(B(x_0, r)) \subset \mathbb{R}^n$, such that C^k is bijective with a C^k inverse, and we have that $\phi_{x_0}(U \cap B(x_0, r)) = \phi_{x_0}(U) \cap \mathbb{R}^{n-1} \times (0, \infty)$

Here we interpret the diffeomorphism as straightening the boundary. The new coordinates given by ϕ_{x_0} are called straightening coordinates.

Definition 2.9.
(i) We say that a function $g : \Gamma \to \mathbb{R}$ is of class C^k and denote $g \in C^k(\Gamma)$ if Γ is C^k and for every $x_0 \in \Gamma$, there exist straightening coordinates $\phi_{x_0} : B(x_0, r) \to \phi_{x_0}(B(x_0, \Gamma))$ such that $g \circ \phi_{x_0}^{-1} \in C^k(\phi_{x_0}(B(x_0, r) \cap \mathbb{R}^{n-1} \times \{0\}))$ in the classical sense. This is independent of choice of ϕ_{x_0}.

(ii) We say that $f \in C^k(\overline{U})$ if $f \in C^k(U)$ and for every $j \in \{0, \ldots, k\}$, $D^j f$ is uniformly continuous in U (which implies that $D^j f$ can be extended as a continuous function on $\overline{U}$).

Remark 2.3.
(i) $f \in C^k(\overline{U})$ is defined even if ∂U is not C^k

(ii) When ∂U is C^k then $f \in C^k(\overline{U}) \iff f \in C^k(U)$ and for each $x_0 \in \partial U$, there exist straightening coordinates ϕ_{x_0} at x_0 such that $f \circ \phi_{x_0}^{-1} \in C^k(\phi_{x_0}(B(x_0, r)) \cap \mathbb{R}^{n-1} \times (0, \infty))$ in the classical sense

Definition 2.10. (Local parametrization of Γ near x_0) $\psi_{x_0} = \phi_{x_0}^{-1}$ and $y_0 = \phi_{x_0}(y_0)$ so that $\psi_{x_0}(y_0) = x_0$ and $\psi_{x_0}|_{\mathbb{R}^{n-1} \times \{0\}}$

Definition 2.11. (Tangent Space) We denote $T_{x_0}\partial U = \text{span}(\partial y_1 \psi_{x_0}(y_1), \ldots, \partial_{y_{n-1}} \psi_{x_0}(y_1))$ as the tangent space of ∂U at x_0.

Definition 2.12. (Outward normal vector) The outward normal vector to ∂U at x_0 is $\nu(x_0)$ which is a unique outward pointing vector such that $|\nu(x_0)| = 1$ and $< \nu(x_0), \partial_{y_i} \psi_{x_0}(y_0) >= 0 \forall i \in [1, n-1]$

Some formulas you can use to find the outer normal vector:

(i) Case (n=3): $\nu(x_0) = \frac{-\partial_{y_1} \psi_{x_0}(y_0) \wedge \partial_{y_2} \psi_{x_0}(y_0)}{|\partial_{y_1} \psi_{x_0}(y_0) \wedge \partial_{y_2} \psi_{x_0}(y_0)|}$

(ii) Case (n=2): $\nu(x_0) = \frac{(v_2 - v_1)}{\sqrt{v_1^2 + v_2^2}}$ when $\partial y_1 \psi_{x_0}(y_0) = (v_1, v_2)$

Tangent Spaces and Outward Normal Vectors Given a Parametrization Examples:

Example 2.6. $U = \{(x,t) \in (0,\infty)^2 : t > 1/x\}$
$\Gamma = \{(x, 1/x) : x \in (0, \infty)\}$
$\psi(x) = (x, 1/x)$
$T(x,t)\partial U = \text{span}(1, \frac{-1}{x^2}) = \text{span}(1, -t^2)$
$\nu(x,t) = \frac{(-t^2, 1)}{\sqrt{1+t^4}}$

Example 2.7. $U = \{(x,y,t) \in \mathbb{R}^2 \times (0,\infty) : t < \sqrt{x^2+y^2}\}$
$\Gamma = \{(x, y, \sqrt{x^2+y^2}) : (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}\}$
$\psi(x,y) = (x, y, \sqrt{x^2+y^2})$
$T_{(x,y,t)}\partial U = \text{span}\{(1, 0, \frac{x}{\sqrt{x^2+y^2}}), (0, 1, \frac{x}{\sqrt{x^2+y^2}})\} = \text{span}\{(1, 0, \frac{x}{t}), (0, 1, \frac{x}{t})\}$
$\nu(x,y,t) = \frac{(1,0,x/t) \wedge (0,1,x/t)}{|(1,0,x/t) \wedge (0,1,x/t)|} = \frac{(-x/t, -y/t, 1)}{\sqrt{(-y/t)^2 + (-x/t)^2 + (1)^2}} = \frac{(-x/t, -y/t, 1)}{\sqrt{2}}$

BVP: $\begin{cases} F(Du,u,x)=0, \text{ in } U \\ u=g, \text{ on } \Gamma \end{cases}$ $\quad F\in C^2(\mathbb{R}^n\times\mathbb{R}\times\overline{U}), g\in C^2(\Gamma), u\in C^2(\overline{U})$

Definition 2.13. (Compatibility Conditions)

$$\begin{cases} z_0=g(x_0) \\ \partial y_i(u\circ\psi_{x_0})(y_0)=<p_0,\partial y_i\psi_{x_0}(y_0)>=\partial y_i(g\circ\psi_{x_0})(y_0) \quad \forall i\in\{1,\dots,n-1\} \\ F(p_0,z_0,x_0)=0 \end{cases}$$

Definition 2.14. (Non-characteristic condition)
$< D_pF(p_0,z_0,x_0),\nu(x_0) >\neq 0$

Example 2.8. (Hamilton-Jacobi)

$$\begin{cases} \partial_t u+\frac{1}{2}|Du|^2=0, \text{ in } U=\{(x,y,t)\in\mathbb{R}^2\times(0,\infty), t<\sqrt{x^2+y^2}\} \\ u(x,y,\sqrt{x^2+y^2})=h(x,y) \text{ on } \Gamma=\{(x,y,\sqrt{x^2+y^2})=(x,y)\in\mathbb{R}^2\backslash\{(0,0)\}\} \end{cases}$$ $F(p_x,p_y,p_t,x,y,z,t)=p_t+\frac{1}{2}(p_x^2+p_y^2)$

Compatibility conditions:

$$\begin{cases} z_0=h(x_0,y_0) \\ p_{x,0}+\frac{x_0}{\sqrt{x_0^2+y_0^2}}=\partial_x h(x_0,y_0) \\ p_{y,0}+\frac{y_0}{\sqrt{x_0^2+y_0^2}}=\partial_y h(x_0,y_0) \\ p_{t,0}+\frac{1}{2}(p_{x,0}^2+p_{y,0}^2)=0 \end{cases}$$

Non-characteristic condition: $<(p_{x,0},p_{y,0},1),(\frac{(-x_0/t_0,-y_0/t_0,1)}{\sqrt{2}})>\neq 0$ as desired.

Case h-0:

$$\begin{cases} z_0=0 \\ p_{x,0}=-\frac{x_0}{t_0}p_{t,0} \\ p_{y,0}=-\frac{y_0}{t_0}p_{t,0} \\ p_{t,0}+\frac{1}{2}(p_{x,0}^2+p_{y,0}^2)=0 \end{cases} \iff \begin{cases} z_0=0 \\ p_{x,0}=-\frac{x_0}{t_0}p_{t,0} \\ p_{y,0}=-\frac{y_0}{t_0}p_{t,0} \\ p_{t,0}=-\frac{1}{2}(\frac{x_0^2+y_0^2}{t_0^2})p_{t,0} \end{cases} \iff (p_{x,0},p_{y,0},p_{t,0})=(0,0,0)$$

Theorem 2.1. (Local Existence and Uniqueness Theorem) Assume that Γ is a C^2 open subset of ∂U, $F\in C^2(\mathbb{R}^n\times\mathbb{R}\times\overline{U})$, $g\in C^2(\Gamma)$, and $(p_0,z_0,x_0)\in\mathbb{R}^n\times\mathbb{R}\times\Gamma$ satisfy both the compatibility conditions and the non-characteristic condition. Then there exists a neighborhood W of x_0 in $U\cup\Gamma$ and a unique solution $u\in C^2(\overline{W})$ of $\begin{cases} F(Du,u,x)=0, \text{ in } W \\ u=g, \text{ on } W\cap\Gamma \end{cases}$ such that $u(x_0)=z_0$ and $Du(x_0)=p_0$

Proof. omitted... □

Remark 2.4. (i) Counter examples to global existence are in A1Q2

(ii) Counter example when the non-characteristic condition isn't satisfied is in A1Q3

(iii) In the quasilinear case, $<b(u,x),Du>=f(u,x)$, the same result holds with $\Gamma\in C^1, b,f\in C^1, g\in C^1, u\in C^1$

(iv) This method and theorem can be applied extended to higher order BVPs

(v) The method of characteristics can also be applied to some wave-type BVPs like in A1Q5

3 Linear Second Order PDEs: The Classical Theory

3.1 The Laplace Equation

For this equation, we assume $n \geq 2$ as the $n = 1$ case is an ODE.

Definition 3.1. (Harmonic)
Any solution $u \in C^2(U)$ of $\Delta u = 0$ is called harmonic (subharmonic if $\Delta u \geq 0$ and superharmonic if $\Delta u \leq 0$)

Case where u is irradially symmetric in $\mathbb{R}^n \backslash \{0\}$:
$u = \hat{u}(r)$ where $r = |x| = \sqrt{\sum_{i=1}^{n} x_i^2} \quad \forall x \in \mathbb{R}^n \backslash \{0\}$

$$\partial x_i u = \frac{x_i}{r}\hat{u}'$$

$$\partial^2 x_i u = (\frac{x_i}{r})^2 \hat{u}'' + (\frac{1}{r} - \frac{x_i^2}{r^3})\hat{u}'$$

$$\Delta u = \hat{u}'' + \frac{n-1}{r}\hat{u}' = 0$$

$$\hat{u}'(r) = \lambda e^{-\int \frac{n-1}{r} dr} = \lambda r^{1-n} \in \mathbb{R}$$

$$\hat{u}(r) = \begin{cases} \lambda ln(r) + \mu, & n = 2 \\ \lambda r^{2-n} + \mu, & n \geq 3 \end{cases}$$

The only harmonic solutions on $\mathbb{R}^n$ that are radially symmetric are the constants.

Recall:
$$\int_{B(x,r)} u = \int_0^r (\int_{\partial B(x,r)} u) dt \quad \text{and} \quad \frac{d}{dr}\int_{B(x,r)} u = \int_{\partial B(x,r)} u$$

Theorem 3.1. (Mean Value Formula)
Let $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\} \subset U$

(i) $\Delta u \geq 0$ in U then $u(x) \leq ⨍_{\partial B(x,r)} u$ and $u(x) \leq ⨍_{B(x,r)} u$

(ii) $\Delta u \leq 0$ in U then $u(x) \geq ⨍_{\partial B(x,r)} u$ and $u(x) \geq ⨍_{B(x,r)} u$

(iii) $\Delta u = 0$ in U then $u(x) = ⨍_{\partial B(x,r)} u$ and $u(x) = ⨍_{B(x,r)} u$

Proof. Let $\phi(r) = ⨍_{\partial B(x,r)} u =$ change of variables (c.o.v) $= \frac{\int_{\partial B(0,1)} r^{n-1} dS(z)}{\int_{\partial B(0,1)} r^{n-1} dS(z)} = ⨍_{\partial B(0,1)} u(x + rz) dS(z)$
Now $\phi'(r) = \frac{d}{dr} ⨍_{\partial B(0,1)} u(x + rz) dS(z) =$ invert c.o.v $= \frac{\int_{\partial B(x,r)} <Du(\hat{z}), \frac{\hat{z}-x}{r}> r^{n-1} dS(\hat{z})}{\int_{\partial B(x,r)} r^{n-1} dS(\hat{z})} = ⨍_{\partial B(x,r) \partial_\gamma u}$
By the divergence theorem, it follows that: $\phi'(r) = ⨍_{B(x,r)} \Delta u \geq 0$ if u is subharmonic.
On the other hand, $\phi(r) \xrightarrow{r \to 0} u(x)$. Indeed:

$$|\phi(r) - u(x)| = | ⨍_{\partial B(x,r)} u(y) dS(y) - u(x)| = | \int_{\partial B(x,r)} (u(y) - u(x)) dS(y)| \leq \sup_{\partial B(x,r)} |u(y) - u(x)| ⨍_{\partial B(x,r)} 1 \to 0$$

by the continuity of u. It follows that $u(x) \leq \phi(r)$. Moreover,

$$\int_{B(x,r)} u = \int_0^r (\int_{\partial B(x,s)} u) ds \geq \int_0^r u(x) (\int_{\partial B(x,s)} 1) ds = u(x) \int_0^r \int_{\partial B(x,s)} 1 ds = u(x) \int_{B(x,r)} 1$$

That is, $u(x) \leq ⨍_{B(x,r)} u$ ∎

Theorem 3.2. (Maximum Principles)
Assume that U is bounded and $u \in C^2(U) \cup C^0(\overline{U})$ be subharmonic in U. Then:

(i) Weak maximum principle: $\max\limits_{\overline{U}} u \leq \max\limits_{\partial U} u$

(ii) Strong maximum principle: if U is connected, then either u is constant or $u < \max\limits_{\overline{U}}$ in U.

Note that the strong maximum principle implies the weak maximum principle. In the superharmonic case, we have the equivalent result but for the minimum.

Proof. It suffices to prove (ii) as (i) follows from (ii) by considering the connected components of U. Define $S = \{x \in U : u(x) = \max\limits_{\overline{U}} u\}$. For each $x \in S$, letting $r > 0$ be small enough so that $B(x,r) \subset U$, the Mean Value Formula gives:

$$\max_{B(x,r)} u \leq \max_{\overline{U}} u = u(x) = \fint_{B(x,r)} u \implies u = \max_{B(x,r)} u \text{ everywhere}$$

Thus $B(x,r) \subset S$. This proves that S is open, it is also a closed set as it can be represented by the inverse function of a singleton, namely $S = u^{-1}(\{\max\limits_{\overline{U}} u\})$.
Since U is connected it follows that $S = \emptyset$ or $S = U$. Thus we have our desired result. □

Theorem 3.3. (Harnack's Inequality)
Let V be an open, bounded, connected subset of U such that $\overline{V} \subset U$. Then there exists a constant C depending only on U and V such that:

$$\sup_V u \leq C \inf_V u$$

for any non-negative harmonic function in U. In particular, $\frac{1}{C}u(y) \leq u(x) \leq Cu(y)$ for all points $x, y \in V$.

Proof. We separate the proof into two cases, first the case with balls and then we generalize this.
Case $V = B(z_0, r)$ and $B(z_0, 4r) \subset U$:
Observe that $B(x_0, r) \subset B(y_0, 3r) \subset B(z_0, 4r) \subset U$.
Indeed, for all $y \in \mathbb{R}^n$:

$$|y - x_0| < r \implies |y - y_0| \leq |y - x_0| + |x_0 - z_0| + |z_0 - y_0| < 3r$$

$$|y - y_0 < 3r \implies |y - z_0| \leq |y - y_0| + |y_0 - z_0| < 4r$$

The Mean Value Formula gives:

$$u(x_0) \int\limits_{B(x_0,r)} 1 = u(x_0)(r)^n \int\limits_{B(0,1)} 1 = \int\limits_{B(x_0,r)} u \leq \int\limits_{B(y_0,3r)} u = u(y_0) \int\limits_{B(y_0,3r)} 1 = u(y_0)(3r)^n \int\limits_{B(0,1)} 1$$

which gives $u(x_0) \leq 3^n u(y_0)$, our desired result.
General case:
Define $r_V = \frac{1}{4}\text{dist}(V, \partial U) = \frac{1}{4}\inf\{|x - y| : x \in V, y \in \partial U\}$
Since $\overline{V}$ is closed and bounded, it is compact, so there exist a finite number N_V of balls $B(z_1, r_V), \ldots, B(z_{N_V}, r_V)$ such that $z_1, \ldots, z_{N_V} \in V$, and $\overline{V} \subset \bigcup\limits_{i=1}^{N_V} B(z_i, r_V)$. Let $i_{x_0}, i_{y_0} \in \{1, \ldots, N_V\}$ be such that $x_0 \in B(z_{i_{x_0}}, r_V)$ and $y_0 \in B(z_{i_{y_0}}, r_V)$. Since V is connected, there exists paths $\gamma : [0,1] \to V$ continuous such that $\gamma(0) = z_{i_{x_0}}$ and $\gamma(1) = z_{i_{y_0}}$.
We then obtain the existence of a finite sequence $(i_j)_{1 \leq j \leq N}$ in $\{1, \ldots, N_V\}$ such that $N \geq N_V$, $i_1 =$

$i_{x_0},\ i_N = i_{y_0},\ \gamma([0,1]) \subset \bigcup_{j=1}^{N} B(z_{i_j}, r_V)$ to ensure that any loops in the path are eliminated, and $B(z_{i_j}, r_V) \cap B(Z_{i_{j+1}}, r_V) \neq 0$.
Now for each $j \in \{1, \dots, N-1\}$,

$$\begin{aligned}\sup_{B(z_{i_j}, rV)} u \leq 3^n \inf_{B(z_{i_j}, r_V)} u \leq 3^n \inf_{B(z_{i_j}, r_V) \cap B(Z_{i_{j+1}}, r_V)} u \leq \dots \leq 3^{N-1} & \sup_{B(z_{i_N}, r_V)} \\ &< 3^{nN} \inf_{B(z_{i_N}, r_V)} u \\ &= 3^{nN} \inf_{B(z_{i_N}, r_V)} u(y_0) \\ &\leq 3^{nN_V} \inf_{B(z_{i_N}, r_V)} u(y_0)\end{aligned}$$

Since $N \leq N_V$. □

Definition 3.2. (Mollifiers)
We call mollifiers any family of functions $(\eta_\epsilon)_{\epsilon>0}$ in $\mathbb{R}^n$ defined as:

$$\eta_\epsilon(x) = \epsilon^{-n}\eta(\epsilon^{-1}(x)) \quad \forall x \in \mathbb{R}^n$$

for some function $\eta \in C^\infty(\mathbb{R}^n)$ such that:

(i) $\eta \equiv 0$ in $\mathbb{R}^n \backslash B(0,1)$

(ii) $\eta \geq 0$ in $B(0,1)$ and $\int_{B(0,1)} \eta = 1$

Example 3.1. Take $\eta(x) = \begin{cases} c\exp(-\frac{1}{1-|x|^2}), & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$

Remark 3.1. There are more general definitions of mollifiers in literature.

Theorem 3.4. (Regularity Theorem) Any harmonic function in U is smooth (C^∞) in U.

Proof. Let u be harmonic in U. Define

$$\eta_\epsilon \star u(x) = \int_{\mathbb{R}^n} \eta_\epsilon(x-y)u(y)dy = \int_{B(x,\epsilon)} \eta_\epsilon(x-y)u(y)dy$$

for all $x \in U_\epsilon = \{y \in U : d(y, \partial U) > \epsilon\}$. Observe that $\eta_\epsilon \star u \in C^\infty(U_\epsilon)$. Indeed,

$$\begin{gathered}|\eta_\epsilon \star u(x+z) - \eta_\epsilon \star u(x) - \int_{\mathbb{R}^n} < D\eta_\epsilon(x-y), z > u(y)dy| \\ = |\int_{\mathbb{R}^n} (\eta_\epsilon(x+z-y) - \eta_\epsilon(x-y) - < D\eta_\epsilon(x-y), z > u(y)dy)| \\ \leq \sup_{y \in B(x,\epsilon) \cup B(x+z,\epsilon)} |\eta_\epsilon(x+z-y) - \eta_\epsilon(x-y) - < D\eta_\epsilon(x-y), z > | \int_{y \in B(x,\epsilon) \cup B(x+z,\epsilon)} |u(y)|dy \\ \leq o(|z|) \int_{B(x,2\epsilon)} |u|\end{gathered}$$

This gives the first derivative, and the higher derivatives will follow by induction.
Now assuming that η_ϵ is radially symmetric about the origin, we can write:

$$\begin{aligned}
\eta_\epsilon \star u(x) &= \int_{B(x,\epsilon)} \eta_\epsilon(x-y)u(y)dy \\
&= \int_0^\epsilon \int_{\partial B(x,s)} \eta_\epsilon(s)u(y)dS(y)ds \\
&= \int_0^\epsilon \eta_\epsilon(s) \int_{\partial B(x,s)} u(y)dS(y)ds \\
&= \int_0^\epsilon \eta_\epsilon(s)u(x) \int_{\partial B(x,s)} dS(y)ds \quad \text{by the mean value theorem} \\
&= u(x)\int_0^\epsilon \int_{\partial B(x,\epsilon)} \eta_\epsilon(s)dS(y)dy \\
&= u(x) \int_{B(x,\epsilon)} \eta_\epsilon \\
&= u(x)
\end{aligned}$$

Hence $u = \eta_\epsilon \star u$ is smooth in U_ϵ. Now, letting $\epsilon \to 0$ we obtain $u \in C^\infty(U)$. □

Definition 3.3. (Fundamental Solution of Laplace's Equation) The function $\phi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ is defined as:

$$\phi(x) = \begin{cases} \frac{1}{2\pi}\log(\frac{1}{|x|}), & n = 2 \\ \frac{|x|^{-(n-2)}}{n(n-2)\alpha(n)}, & n \geq 3 \end{cases}$$

where $\alpha(n) = \int_{B(0,1)} 1$ is the volume of the unit ball in $\mathbb{R}^n$. This function is the fundamental solution of $-\Delta$.

Theorem 3.5. (Green's Representation Formula I)
Assume that U is bounded and ∂U is C^1. Then for every $u \in C^2(\overline{U})$ and $x \in U$,

$$u(x) = \int_{\partial U} (\phi_x \partial_\nu u - u\partial_\nu \phi_x) - \int_U \phi_x \Delta u$$

where $\phi_x = \phi(x-y)$

Proof. Let $\epsilon > 0$ be small enough such that $B(x,\epsilon) \subset U$. Then apply the divergence theorem to:

$$\begin{aligned}
\int_{\partial U \cup \partial B(x,\epsilon)} (\phi_x \partial_\nu u - u\partial_\nu \phi_x) &= \int_{U\backslash B(x,\epsilon)} \text{div}(\phi_x \Delta u - uD\phi_x) \\
&= \int_{U\backslash B(x,\epsilon)} \phi_x\Delta u + < D\phi_x, Du > -u\Delta\phi_x - < D\phi_x, Du > \\
&= \int_{U\backslash B(x,\epsilon)} \phi_x\Delta u - u\Delta\phi_x \\
&= \int_{U\backslash B(x,\epsilon)} \phi_x\Delta u \quad \text{by the harmonicity of } \phi_x
\end{aligned}$$

On $\partial B(x,\epsilon)$,

$$\phi_x(y) = \begin{cases} \frac{1}{2\pi}\ln(\epsilon), & n=2 \\ \frac{\epsilon^{-(n-2)}}{n(n-2)\alpha(n)}, & n \geq \end{cases}$$

and

$$\partial_\nu \phi_x(y) = \begin{cases} \frac{1}{2\pi\epsilon}, & n=2 \\ \frac{-(n-2)\epsilon^{-(n-1)}}{n(n-2)\alpha(n)}, & n \geq 3 \end{cases}$$

which gives

$$\int\limits_{\partial B(x,\epsilon)} = \begin{cases} \epsilon\ln(\epsilon) ⨍_{\partial B(x,\epsilon)} \partial_\nu u, & n=2 \\ \frac{\epsilon}{n-2} ⨍_{\partial B(x,\epsilon)} \partial_\nu u, & n \geq 3 \end{cases}$$

and

$$\int_{\partial B(x,\epsilon)} \phi_x \partial_\nu u = \begin{cases} \epsilon\log(\epsilon) ⨍_{\partial B(x,\epsilon)} \partial_\nu u, & n=2 \\ \frac{\epsilon}{n-2} ⨍_{\partial B(x,\epsilon)} \partial_\nu u, & n \geq 3 \end{cases} = o\left(⨍_{\partial B(x,\epsilon)} \partial_\nu u \right) \to 0 \text{ as } \epsilon \to 0$$

Since

$$| ⨍_{\partial B(x,\epsilon)} \partial_\nu u| \leq ⨍_{\partial B(x,\epsilon)} |\partial_\nu u| \leq ⨍_{\partial B(x,\epsilon)} |Du| \to |Du(x)|$$

by continuity of u. Moreover,

$$\left| \int_{U\setminus B(x,\epsilon)} \phi_x \Delta u - \int_U \phi_x \Delta u \right| = \left| \int_{B(x,\epsilon)} \phi_x \Delta u \right| \leq \max_{B(x,\epsilon)} |\Delta u| \int_{B(x,\epsilon)} \phi_x$$

Where we have that:

$$\int_{B(x,\epsilon)} \phi_x = \begin{cases} \int_0^\epsilon \log(\frac{1}{r})dr, & n=2 \\ \frac{1}{n-2}\int_0^\epsilon r^{2-n}r^{n-1}dr, & n \geq 3 \end{cases} \to 0 \text{ as } \epsilon \to 0$$

Putting all of this together and letting $\epsilon \to 0$ we get our formula. □

3.2 Poisson's Equation

Definition 3.4. (Poisson's Equation)
We define Poisson's Equation by $\begin{cases} -\Delta u = f, & \text{in } U \\ u = g, & \text{on } \partial U \end{cases}$ for some functions $f \in C^0(U)$ and $g \in C^0(\partial U)$.

Theorem 3.6. (Poisson's Formula in $\mathbb{R}^n$)
Given $f \in C_c^2(\mathbb{R}^n)$, which is the set of C^2 functions with compact support in $\mathbb{R}^n$. Then the function $u : \mathbb{R}^n \to \mathbb{R}^n$ defined by $u(x) = \int\limits_{\mathbb{R}^n} \phi_x f$ is a solution of Poisson's Equation.

Proof. By change of variables we obtain $u(x) = \int_{\mathbb{R}^n} \phi(x-y)f(y)dy = \int_{\mathbb{R}^n} \phi(y)f(x-y)dy$
By differentiating with respect to x we obtain that $u \in C^2(\mathbb{R}^n)$ and:

$$\forall j \in 1,2 \quad D^j u = \int_{\mathbb{R}^n} \phi(y) D^j f(x-y)dy$$

Indeed, the same arguments for $\eta_\epsilon \star u$ from the previous regularity theorem still applies since $f \in C^2(\mathbb{R}^n)$ and $\int_A \phi < \infty$ for any compact set $A \subset \mathbb{R}^n$. This was verified for $A = B(0,\epsilon)$ in the proof of Greens

Formula I. In particular,

$$\begin{aligned}\Delta u &= \int_{\mathbb{R}^n} \phi(y)\Delta f(x-y)dy \\ &= \int_{\mathbb{R}^n} \phi(x-y)f(y)dy \\ &= \int_{\mathbb{R}^n} \phi_x \Delta f \\ &= -f(x)\end{aligned}$$

The last equality holds by applying Green's Representation Formula I in $B(0,R)$ with R large enough so that $f = 0$ in $\mathbb{R}^n \setminus B(0,R)$. This gives the desired result that $-\Delta u = f$ in $\mathbb{R}^n$. $\square$

Definition 3.5. (Green's Function)
Let $D = \{(x,x) : x \in U\}$. We define the Green's Function of $-\Delta$ in U as any function $G : U^2 \setminus D \to \mathbb{R}$ defined by:

$$G(x,y) = \phi(x-y) - \psi(x-y) \quad \forall (x,y) \in U^2 \setminus D$$

where for each $x \in U$, $\psi_x = \psi(x,\cdot) \in C^2(\overline{U})$ is a solution of:

$$\begin{cases} \Delta\psi_x = 0, & \text{in } U \\ \psi_x = \phi_x, & \text{on } \partial U \end{cases} \text{ so that } \begin{cases} \Delta G_x = 0, & \text{in } U \setminus \{x\} \\ G_x = 0, & \text{on } \partial U \end{cases}$$

Example 3.2. Let $U = \mathbb{R}^{n-1} \times (0,\infty)$. Then $G(x,y) = \phi(x-y) - \phi(\tilde{x}-y)$ where $\tilde{x} = (x_1, \ldots, x_{n-1}, -x_n)$.

Example 3.3. Let $U = B(0,R)$. Then:

$$G(x,y) = \phi(x-y) - \underbrace{\phi(\frac{R}{|x|}x - \frac{|x|}{R}y)}_{\frac{|x|}{R}(\frac{R^2}{|x|^2}x - y)}$$

$$\begin{cases} \forall y \in U, & \Delta[\phi(\frac{R}{|x|}x - \frac{|x|}{R}y)] = (-\frac{|x|}{R})^2 \Delta\phi(\frac{R}{|x|}x - \frac{|x|}{R}y) = 0 \\ \forall y \in \partial U, & |\frac{R}{|x|}x - \frac{|x|}{R}y|^2 = |\frac{R}{|x|}x|^2 + |\frac{|x|}{R}y|^2 - 2 < \frac{R}{|x|}x, \frac{|x|}{R}y >= R^2 + |x|^2 - 2 < x, y >= |x-y|^2 \end{cases}$$

Remark 3.2. The question of existence of Green's functions in more general bounded domains (and more general operators) will be addressed future chapters.

Theorem 3.7. (Green's Representation Formula II) Assume that U is bounded and ∂U is C^1. Let G be a Green's Function of $-\Delta$ in U. Then for every $u \in C^2(\overline{U})$ and $x \in U$,

$$u(x) = -\int_{\partial U} g\partial_\nu G_x - \int_U G_x \Delta u$$

In particular, if $u \in C^2(\overline{U})$ and it is a solution of Poisson's Equation for some $f \in C^0(U)$ and $g \in C^0(\partial U)$ then

$$u(x) = -\int_{\partial U} g\partial_\nu G_x + \int_U fG_x$$

Proof. The divergence theorem gives us:

$$\int_{\partial U} \psi_x \partial_\nu u - u\partial_\nu \psi_x = \int_U \text{div}(\psi_x Du - uD\psi_x) = \int_U \psi_x \Delta u$$

By summing with the Green's Representation Formula I, we obtain:

$$\underbrace{\int_{\partial U} G_x \partial_\nu u}_{=0 \text{ on } \partial U} \underbrace{-u\partial_\nu G_x}_{=-g\partial_\nu G_x \text{ on } \partial U} = \int_U G_x \Delta u + u(x)$$

Thus we obtain:

$$u(x) = -\int_{\partial U} g\partial_\nu G_x - \int_U G_x \Delta u$$

□

Remark 3.3. (i) The converse is not true in general.
Indeed, $\int_U fG_x$ is not even C^2 in general. We can prove that it is C^2 when f is Holder's continuous.

(ii) In case $U = B(0,R)$ and $f = 0$ we obtain Poisson's formula for balls: $u(x) = \frac{R^2-|x|^2}{n\alpha(n)R} \int_{\partial B(0,R)} \frac{g(y)}{|x-y|^n} dS(y)$

Theorem 3.8. (Uniqueness of Poisson's Equation)
Assume that U is bounded. Then there exists at most one solution $u \in C^2(U) \cap C^0(\overline{U})$ of

$$\begin{cases} -\Delta u = f, \text{ in } U \\ u = g, \text{ on } \partial U \end{cases}$$

Proof. Assume there are two solutions u_1 and u_2. Let $u = u_1 - u_2$, which solves $-\Delta u = 0$ in U and $u = 0$ on ∂U. By the weak maximum principle, it follows that $\max_{\overline{U}} u = \min_{\overline{U}} u = 0$ in U. Hence $u = 0$ in U and $u_1 = u_2$. □

We will also provide an alternate proof based on the energy method.

Proof. (Energy Method)
We have that U is C^1 and $u \in C^2(\overline{U})$. We then multiply $\Delta u = 0$ by u and integrate by parts to obtain:

$$0 = \int_U u\Delta u = \underbrace{\int_{\partial U} u\partial_\nu u}_{=0 \text{ since } u=0 \text{ on } \partial U} - \int_U < Du, Du >= -\int_U |Du|^2$$

Since $|Du|^2 \geq 0$ in u, it follows that $Du = 0$ in U. Thus u is constant and since $u = 0$ on ∂U, it follows that $u = 0$ in U. □

Remark 3.4. $u \mapsto \int_U |Du|^2$ is referred to as the energy of $-\Delta$ in U.

Properties 3.1. Assume that U is bounded and let G be a Green's Function of $-\Delta$ in U. Then:

(i) Uniqueness: G is the unique Green's Function

(ii) Positivity: $\forall (x,y) \in U^2 \setminus D, G(x,y) \geq 0$. Moreover, if U is connected then $G(x,y) > 0$.

(iii) Symmetry: If ∂U is C^2 $\forall (x,y) \in U^2 \setminus D$, then $G(x,y) = G(y,x)$

Proof. (i) Assume that there are two Green's Functions G_1 and G_2. Define $G = G_1 - G_2 = \phi + \psi_1 - \phi - \psi_2$ where $G_1 = \phi + \psi_1$ and $G_2 = \phi + \psi_2$. Then $G_x \in C^2(\overline{U})$ solves $-\Delta G_x = 0$ in U and $G_x = 0$ on ∂U. Hence by the uniqueness theorem, $G = 0$ in U therefore $G_1 = G_2$.

(ii) Let $\epsilon > 0$ be small enough so that $B(x,\epsilon) \subset U$. Since $\lim_{y\to x} G_x(y) = \infty$, we obtain that $G_x(y) > 0$ on $\partial B(x,\epsilon)$ for small $\epsilon > 0$. On the other hand, $G_x = 0$ on ∂U, so by the maximum principle, it gives that $G_x \geq 0$ in $U \setminus B(x,\epsilon)$. And if U is connected, then either G_x is constant on $\partial B(x,\epsilon)$ or $G_x > 0$ in $U \setminus B(x,\epsilon)$. However, G_x is not constant on $\partial(U \setminus B(x,\epsilon))$ since $G_x = 0$ on ∂U and $G_x > 0$ on $\partial B(x,\epsilon)$, hence $G_x > 0$ in $U \setminus B(x,\epsilon)$ when U is connected. Letting $\epsilon \to 0$, we obtain that $G_x \geq 0$ in $U \setminus \{x\}$ and if U is connected, then $G_x > 0$ in $U \setminus \{x\}$.

(iii) Let $\epsilon > 0$ be small enough so that $B(x,\epsilon) \subset U$ and $B(y,\epsilon) \subset U$ such that $B(x,\epsilon) \cap B(y,\epsilon) = \emptyset$. Then by the divergence theorem, we obtain:

$$\int\limits_{U\setminus(B(x,\epsilon)\cup(B(y,\epsilon)))} \operatorname{div} \underbrace{(G_x DG_y - G_y DG_x)}_{=G_x\Delta G_y - G_y \Delta G_x = 0} = \int\limits_{\partial U\setminus(B(x,\epsilon)\cup(B(y,\epsilon)))} (G_x\partial_\nu G_y - G_y \partial_\nu G_x)$$

Since $G_x = G_y = 0$ on ∂U, it follows that:

$$\int\limits_{\partial(B(x,\epsilon)\cup(B(y,\epsilon)))} (G_x\partial_\nu G_y - G_y \partial_\nu G_x) = 0$$

Moreover, by same reasoning as in the proof of the Green's Representation Formula I,

$$\int\limits_{\partial B(x,\epsilon)} (G_x\partial_\nu G_y) \to 0 \text{ as } \epsilon \to 0$$

$$\int\limits_{\partial B(x,\epsilon)} (G_y\partial_\nu G - x) \to 0 \text{ as } \epsilon \to 0$$

$$\int\limits_{\partial B(y,\epsilon)} (G_x\partial_\nu G_y) \to G_x(y) \text{ as } \epsilon \to 0$$

$$\int\limits_{\partial B(y,\epsilon)} (G_y\partial_\nu G - x) \to G_y(x) \text{ as } \epsilon \to 0$$

Thus $G_x(y) = G_y(x)$ for all $x, y \in U$.

□

3.3 The Heat Equation

Definition 3.6. (Heat Equation)
$\begin{cases}\partial_t u = \Delta u, \text{ in } U \times I\end{cases}$ where U is an open subset of $\mathbb{R}^n$ and I is an interval of $\mathbb{R}$.

Definition 3.7. (Fundamental Solution of the Heat Equation)
We call the fundamental solution of the heat equation the function $\phi : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ as:

$$\phi(x,t) = \frac{1}{(4\pi t)^{n/t}} e^{\frac{-|x|^2}{4t}} \quad \forall (x,t) \in \mathbb{R}^n \times (0,\infty)$$

Proposition 3.1. ϕ is a solution of

$$\begin{cases}\partial_t u = \Delta u, \text{ in } \mathbb{R}^n \times (0,\infty) \\ u(\cdot,0) = 0, \text{ in } \mathbb{R}^n \times \{0\}\end{cases}$$

and $\int_{\mathbb{R}^n} \phi(\cdot,t) = 1$ for all $t \in (0,\infty)$.

Proof. Clearly $\phi \in C^\infty(\mathbb{R}^n \times (0,\infty))$ and it is not difficult to see that $\phi \to 0$ as $t \to 0$ $\forall x \neq 0$.

$$\partial_t \phi - \Delta\phi = \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4t}} (\frac{|x|^2}{4t^2} - \frac{n}{2t}) - \frac{1}{(4\pi t)^{n/2}} \mathrm{div}(-e^{\frac{-|x|^2}{4t}} \frac{x}{2t}) = 0$$

And we have that: $\int_{\mathbb{R}^n} \phi(x,t)dx = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{-|x|^2}{4t}} dx = 1$ □

Definition 3.8. For every subset U of $\mathbb{R}^n$ and interval I in $\mathbb{R}$, we have that:

$$C_1^2(U \times I) = \{u \in C^1(U \times I) : D_x^2 u \text{ exists and is continuous at all points in } U \times I\}$$

$$C_1^2(\overline{U} \times \overline{I}) = \{u \in C_1^2(U \times I : u, \partial_t u, D_x u, D_x^2 u \text{ are uniformly continuous in bounded subsets of } U \times I)\}$$

We say that u is a solution of $\partial_t u = \Delta u + f$ in $U \times I$ if $u \in C_1^2(U \times I)$ and the equation is satisfied at all points in $U \times I$.

Definition 3.9. For every point $(x,t) \in \mathbb{R}^n \times \mathbb{R}, r > 0$, the sets

$$\begin{aligned} E(x,t,r) &= \{(y,s) \in \mathbb{R}^n \times \mathbb{R} : s < t \text{ and } \phi(x-y,t-s) > r^{-n}\} \\ &\iff \frac{1}{(4\pi(t-s))^{n/2}} e^{\frac{-|x-y|^2}{4(t-s)}} > r^{-n} \\ &\iff |x-y|^2 < 2n(t-s)\ln(\frac{r^2}{4\pi(t-s)}) \end{aligned}$$

and $\overline{E(x,t,r)} = \{(y,s) \in \mathbb{R}^n \times \mathbb{R} : s \leq t \text{ and } \phi(x-y,t-s) \geq r^{-n}\}$

Theorem 3.9. (Mean Value Formula)
Let $u \in C_1^2(\overline{E(x,t,r)})$.

(i) If $\partial_t u \leq \Delta u$ in $E(x,t,r)$ then $u(x,t) \leq \frac{1}{4r^n} \int \int_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}$

(ii) If $\partial_t u \geq \Delta u$ in $E(x,t,r)$ then $u(x,t) \geq \frac{1}{4r^n} \int \int_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}$

(iii) If $\partial_t u = \Delta u$ in $E(x,t,r)$ then $u(x,t) = \frac{1}{4r^n} \int \int_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}$

Proof. It suffices to prove (i) as (ii) and (iii) follow from (i). Without loss of generality we may assume that $(x,t) = (0,0)$. Moreover, as a first step we assume that $u \in C^2(E(x,t,r))$.
Define $\phi(r) = \frac{1}{4r^n} \int \int_{E(0,0,r)} u(y,s) \frac{|y|^2}{s^2} dyds$. We want to show that that $\phi'(\rho) \geq 0 \quad \forall \rho \in (0,r)$ and $\phi(\rho) \to u(0,0)$ as $\rho \to 0$. By change of variables, $\phi(\rho) = \frac{1}{4} \iint_{E(0,0,1)} u(\rho y, \rho^2 s) \frac{|y|^2}{s^2} dyds$. Then:

$$\begin{aligned} \phi'(\rho) &= \frac{1}{4} \iint_{E(0,0,1)} < D_y u(\rho y, \rho^2 s) > \frac{|y|^2}{s^2} + 2\rho \partial_s u(\rho y, \rho^2 s) \frac{|y|^2}{s} dyds \\ &= \frac{1}{4\rho^{n+1}} \iint_{E(0,0,\rho)} (< D_y u(y,s), y > + 2\partial_s u(y,s)) \frac{|y|^2}{s} dyds \end{aligned}$$

Introduce the function $\psi(y,s) = \frac{|y|^2}{s} + 2n\ln(\frac{-\rho}{4\pi s})$ so that $\psi = 0$ on $\partial E(0,0,\rho) \setminus \{(0,0)\}$. Then we have that $D_y\psi(y,s) = \frac{2y}{s}$ and $\partial_s\psi(y,s) = \frac{-|y|^2}{s^2} - \frac{2n}{s}$.
It follows that:

$$\begin{aligned} y\frac{|y|^2}{s^2} &= -\partial_s\psi(y,s)y - 2n\frac{y}{s} \\ &= -\partial_s\psi(y,s)y - nD_y\psi(y,s) \end{aligned}$$

and $2\frac{|y|^2}{s} =< D_y\psi(y,s), y >$, which gives:

$$\phi'(\rho) = \frac{1}{4\rho^n}\iint_{E(0,0,\rho)} (-< D_yu(y,s), \partial_s\psi(y,s) + nD_y\psi(y,s) > +\partial_s u(y,s) \underbrace{< D_y\psi(y,s), y >}_{=\mathrm{div}(\psi(y,s),y)-n\psi(y,s)})dyds$$

By integration by parts:

$$\begin{aligned} \iint\limits_{E(0,0\rho)} < D_yu, \partial_s\psi(y) > dyds &= -\iint\limits_{E(0,0,\rho)} < D_y\partial_s u, \psi(y,s)y > \\ &= \iint\limits_{E(0,0,\rho)} \partial_s u\,\mathrm{div}(\psi_y(y,s))dyds \end{aligned}$$

Which gives:

$$\phi'(\rho) = \frac{-n}{4\rho^{n+1}}\iint_{E(0,0,\rho)} (< D_y\partial_s u, \psi(y,s)y > +\partial_s u\psi)dyds$$

On the other hand, by continuous of u,

$$\phi(\rho) = \frac{1}{4}\iint u(\rho y, \rho^2 s)\frac{|y|^2}{s^2}dyds \to \frac{u(0,0)}{4}\iint\limits_{E(0,0,\rho)} \frac{|y|^2}{s^2}dyds$$

$$\iint\limits_{E(0,0,1)} \frac{|y|^2}{s^2}dyds = n\alpha(n)\int_{-1/4\pi}^{0}\int_0^{-2n\ln(\frac{-1}{4\pi s})} \rho^{n+1}d\rho ds$$

Now we consider the case where $u \in C_1^2(\overline{E(0,0,r)})$. Let $(\eta_\epsilon)_{\epsilon>0}$ be a family of mollifiers in $\mathbb{R}^{n+1}$ and observe $\partial_t(\eta_\epsilon \star u) = \eta_\epsilon \star \partial_t u$ and $\Delta(\eta_\epsilon \star u) = \eta_\epsilon \star \Delta u$. So that if $\partial_t u \leq \Delta u$ gives $\partial_t(\eta_\epsilon \star u) \leq \Delta(\eta_\epsilon \star u)$. Moreover, $\eta_\epsilon \star u$ is smooth, hence:

$$\eta_\epsilon \star u(0,0) = \frac{1}{4\rho^n}\iint_{E(0,0,\rho)} \eta_\epsilon \star u(y,s)\frac{|y|^2}{s^2}dyds \quad \forall \rho \in (0,r) \quad (\star)$$

for $\epsilon > 0$ small enough. On the other hand, $\eta_\epsilon \star u \to u$ uniformly in $E(0,0,\rho)$. Indeed,

$$\begin{aligned} |\eta_\epsilon \star u(x,t) - u(x,t)| &= |\iint_{B((x,t),\epsilon)} \eta_\epsilon(x-y,t-s)(u(y,s) - u(x,t))dyds| \\ &\leq \sup_{(y,s)\in B((x,t),\epsilon)} |u(y,s) - u(x,t)| \iint_{B((x,t),\epsilon)} \eta_\epsilon(x-y,t-s)dyds \to 0 \text{ as } \epsilon \to 0 \end{aligned}$$

uniformly in $(x,t) \in E(0,0,\rho)$. Therefore, by letting $\epsilon \to 0$ in $(\star)$ and then $\rho \to r$ in $(\star)$. We obtain $u(0,0) \leq \frac{1}{4r^n}\iint_{E(0,0,r)} u(y,s)\frac{|y|^2}{s^2}dyds$ □

Theorem 3.10. (Maximum Principles)
Assume that U is bounded and assume that $u \in C_1^2(U_T) \cap C^0(\overline{U_T})$, where $U_T = U \times (0,T]$ such that $\partial_t u \leq \Delta u$.

(i) Weak maximum principle:
$$\max_{\overline{U_T}} u = \max_{\Gamma_T} u$$
where $\Gamma_T = \overline{U_T} \setminus U_T = \partial U \times (0,T] \cup (U \times \{0\})$.

(ii) Strong maximum principle: If U is connected, and if there exists $(x_0, t_0) \in U_T$ such that $u(x_0,t_0) = \max_{\overline{U_T}} u$ then u is constant in U_{t_0} (that is, at time $t = 0$).

Proof. By considering connected components of U, (ii) implies (i). Thus it suffices to prove (ii).
Let $x_1 \in U$ and $t_1 \in (0, t_0)$. Since U is connected and open in $\mathbb{R}^n$, there exists a continuous path $\gamma : [0,1] \to U$ and joining the points such that $\gamma(0) = x_0, \gamma(1) = x_1$ and $\gamma([0,1])$ is the union of straight line segments. Then define $\tilde{\gamma} : [0,1] \to U \times [t_1, t_0]$ such that $s \mapsto (\gamma(s), (1-s)t_0 + st_1)$. Define:
$$\overline{S} = \max\{s \in [0,1] : u(\tilde{\gamma}(s')) = \max_{\overline{U_T}} u \forall s' \in [0,s]\}$$
By continuity of u, we have that $u(\widetilde{\gamma(\overline{S}}))= \max_{\overline{U_T}} u$. By appling the Mean Value Formula, we obtain for small $r > 0$, that:
$$\frac{1}{4r^n} \iint_{E(\gamma(\tilde{(\overline{S}})),r)} (\max_{\overline{U_T}} u) \frac{|y-x|^2}{(\overline{s}-\overline{t})^2} dyds = \max_{\overline{U_T}} u = u(\widetilde{\gamma(\overline{s})}) \leq \frac{1}{4r^n} \iint_{E(\widetilde{\gamma(\overline{s})},r)} u(y,s) \frac{|y-x|^2}{(s-\overline{t})^2} dyds$$
where $\widetilde{\gamma(\overline{s})} = (\overline{x}, \overline{t})$. Hence we obtain that $u = \max_{\overline{U_T}} u$ on $E(\widetilde{\gamma(\overline{s})}, r)$. Observe that $\widetilde{\gamma(\overline{s}+\epsilon)} \in E(\widetilde{\gamma(\overline{s})}, r)$ for $\epsilon > 0$ small, hence we obtain a contradiction with the definition of $\overline{s}$ unless if $\overline{s} = 1$, which means that $\widetilde{\gamma(\overline{s})} = (x_1, t_1)$, meaning that $u(x_1, t_1) = \max_{\overline{U_T}} u$. This proves that $u = \max_{\overline{U_T}} u$ in $U \times (0, t_0)$. By continuity, this is true in U_{t_0}. □

Corollary 3.1. (Uniqueness)
Assume that U is bounded. Then there exists at most one solution $u \in C_1^2(U_T) \cap C^0(\overline{U_T})$ of:
$$\begin{cases} \partial_t u = \Delta u + f & \text{in } U \times (0,T] \\ u = g & \text{on } \partial U \times (0,T] \\ u(\cdot, h) = h & \text{in } U \end{cases}$$
Proof. Assume that u_1 and u_2 are two solutions. Then $u = u_1 - u_2$ is a solution of $\partial_t u = \Delta u$ in $U \times (0,T]$ and $u = 0$ on $\partial U \times (0,T]$. By the strong maximum principle, $u = 0$ in $U \times (0,T]$. Hence $u_1 = u_2$. □

Proof. (Alternate Energy Proof) This is the case when ∂U is C^1 and $u \in C_1^2(\overline{U_T})$:
$$E(t) = \int_U u(x,t)^2 dx$$
Then we differentiate with respect to t:
$$E'(t) = 2\int_U u(x,t)\partial_t u dx = 2\int_U u(x,t)\Delta u(x,t) dx$$
We now integrate by parts using the Divergence Theorem:
$$E'(t) = -2\int_U |\nabla u(x,t)|^2 dx + 2\int_{\partial U} u(x,t)\partial_\nu u(x,t) dS(x) \leq 0$$
Hence we obtain that $E(t) \leq E(0) = 0$, and since $u(x,t)^2 \geq 0$, it follows that $u(x,t) = 0 \quad \forall (x,t) \in U_T$. □

Remark 3.5. The function $u \mapsto \{t \mapsto E(u)(t) - \int_U u(x,t)^2 dx\}$ is sometimes referred to as the energy of $\partial_t - \Delta$ in U.

Theorem 3.11. (Harnack's Inequality for the Heat Equation)
Let $t_1, t_2 \in (0,T]$ be such that $t_1 < t_2$ and V an open bounded connected subset of U such that $\overline{V} \subset U$. Then $\exists c > 0$ a constant depending only on t_1, t_2, V and U such that $\sup_V u(\cdot, t_1) \leq c \inf_V u(\cdot, t_2) \quad \forall$ non-negative solutions

Proof. Adapt the proof of Harnack's Inequality for $\Delta u = 0$ in U, by considering a path of straight line segments. □

Theorem 3.12. (Duhamel's Principle)
Let $f : U \times (0,T] \to \mathbb{R}, g : \partial U \times (0,T] \to \mathbb{R}$, and $h : U \to \mathbb{R}$. Let $v_0 \in C_1^2(U_T)$ be a solution of:

$$\begin{cases} \partial_t u = \Delta u \quad \text{in } U \times (0,T] \\ u = g \quad \text{on } \partial U \times (0,T] \\ u(\cdot, 0) = h \quad \text{in } U \end{cases}$$

and $v \in C^0(\{(x, t-s) : x \in \overline{U}, 0 < t \leq T, 0 \leq s \leq t\})$ be such that for each $s \in (0,T)$, $v_s = v(\cdot,\cdot,s) \in C_1^2(U_{T-S})$ a solution of:

$$\begin{cases} \partial_t v_s = \Delta v_s \quad \text{in } U \times (0, T \setminus S] \\ v_s = 0 \quad \text{on } \partial U \times (0, T \setminus S] \\ v_s(\cdot, 0) = f(\cdot,\cdot,s) \quad \text{in } U \end{cases}$$

Assume moreover that for every compact subset K of $\{(x, t-s) : x \in U, 0 < t \leq T, 0 \leq s \leq t\}$, $\partial_t v_s(x,t), D_x v_s(x,t), D_x^2 v_s(x,t)$ are uniformly continuous in $K \cap \bigcup_{0<s<T}(U_{T\setminus S} \times \{s\})$. Then the function $u : U_T \to \mathbb{R}$ defined as

$$u(x,t) = v_0(x,t) + \int_0^t v_s(x, t-s) ds \quad \forall (x,t) \in U_T$$

is a solution of:

$$\begin{cases} \partial_t u = \Delta u + f \quad \text{in } U \times (0,T] \\ u = g \quad \text{on } \partial U \times (0,T] \\ u(\cdot, 0) = h \quad \text{in } U \end{cases}$$

Proof. The regularity of U follows from the assumptions above. Now, $\forall (x,t) \in U \times [0,T]$,

$$\partial_t u(x,t) \partial_t v_0(x,t) + v_t(x,0) + \int_0^t \partial_t v_s(x,t-s) ds = \Delta u(x,t) + f(x,t)$$

Now, $\forall (x,t) \in \partial U \times [0,T]$,

$$u(x,t) = v_0(x,t) + \int_0^t v_s(x,t-s) ds = g(x,t)$$

and $\forall x \in U$, $u(x,0) = v_0(x,t) = h(x)$ as desired. □

Theorem 3.13. Let $h \in C^0(\mathbb{R}^n)$ and $f \in C_1^2(\mathbb{R}^n \times [0,T])$ be such that:

$$\sup_{\mathbb{R}^n} |h| + \sup_{\mathbb{R}^n \times [0,T]} (|f| + |\partial_t f| + |D_x f| + |D_x^2 f|) < \infty$$

Then the function $u : \mathbb{R}^n \times (0,T]$ defined by:

$$u(x,t) = \int_{\mathbb{R}^n} \phi(x-y,t) h(y) dy + \int_0^t \int_{\mathbb{R}^n} \phi(x-y, t-s) f(y,s) dy ds$$

is a solution of:

$$\begin{cases} \partial_t u = \Delta u + f & \text{in } \mathbb{R}^n \times (0,T] \\ u(\cdot,0) = h & \text{on } \mathbb{R}^n \end{cases}$$

Proof. The regularity assumptions from Duhamel's Principle are not applicable here, as we are missing some regularity conditions. We will proceed by a direct proof. By change of variables:

$$u(x,t) = \int_{\mathbb{R}^n} \phi(x-y,t)h(y)dy + \int_0^t \int_{\mathbb{R}^n} \phi(y,s)f(x-y,t-s)dyds$$

By differentiating and using the assumptions on h and f, and also the fact that $\int_{\mathbb{R}^n} \phi(y,s)dy = 1$ and $\int_{\mathbb{R}^n} |\partial_t \phi| < \infty$, $\int_{\mathbb{R}^n} |D_x(\phi)| < \infty$, and $\int_{\mathbb{R}^n} |D_x^2 \phi| < \infty$.
We obtain that $u \in C_1^2(U_T)$ and

$$\partial_t u(x,t) = \int_{\mathbb{R}^n} \partial_t \phi(x-y,t)h(y)dy + \int_{\mathbb{R}^n} \phi(y,t)f(x-y,0)dy + \int_0^t \int_{\mathbb{R}^n} \phi(y,s)\partial_t f(x-y,t-s)dyds$$

and same $\forall j \in \{1,2\}$:

$$D_x^j u(x,t) = \int_{\mathbb{R}^n} D_x^j \phi(x-y,t)h(y)dy + \int_0^t \int_{\mathbb{R}^n} \phi(y,s)D_x^j f(x-y,t-s)dyds$$

In particular, since $\partial_t \phi = \Delta \phi$ we obtain:

$$\partial_t u(x,t) - \Delta u(x,t) = \int_{\mathbb{R}^n} \phi(y,t)f(x-y,0)dy + \int_0^t \int_{\mathbb{R}^n} \phi(y,s)(\partial_t f - \Delta f)(x-y,t-s)dyds$$

Integrate by parts on $B(0,\frac{1}{\epsilon}) \times [\epsilon, t]$:

$$\begin{aligned} \int_\epsilon^t \int_{B(0,\frac{1}{\epsilon})} \phi(y,s)\partial_t f(x-y,t-s)dyds = &\int_\epsilon^t \int_{B(0,\frac{1}{\epsilon})} \partial_t \phi(y,s)f(x-y,t-s)dyds \\ &+ \int_{B(0,\frac{1}{\epsilon})} \phi(y,\epsilon)f(x-y,t-\epsilon)dy - \int_{B(0,\frac{1}{\epsilon})} \phi(y,t)f(x-y,0)dy \end{aligned}$$

We can rewrite this as:

$$\begin{aligned} \int_{B(0,\frac{1}{\epsilon})} \phi(y,t)f(x-y,0)dy + \int_\epsilon^t \int_{B(0,\frac{1}{\epsilon})} \phi(y,s)\partial_t f(x-y,t-s)dyds = &\int_\epsilon^t \int_{B(0,\frac{1}{\epsilon})} \partial_t \phi(y,s)f(x-y,t-s)dyds \\ &+ \int_{B(0,\frac{1}{\epsilon})} \phi(y,\epsilon)f(x-y,t-\epsilon)dy \end{aligned}$$

Integrate by parts on the same domain with respect to the Laplacian:

$$\begin{aligned} \int_\epsilon^t \int_{B(0,\frac{1}{\epsilon})} \phi(y,s)\Delta f(x-y,t-s)dyds = &\int_\epsilon^t \int_{B(0,\frac{1}{\epsilon})} \Delta \phi(y,s)f(x-y,t-s)dyds \\ &+ \int_{\partial B(0,\frac{1}{\epsilon})} \phi(y,s)\partial_\nu f(x-y,t-s) - f(x-y,t-s)\partial_\nu \phi(y,s)dS(y)ds \end{aligned}$$

But some of the terms cancel out. Moreover:

$$\Big| \iint\limits_{\mathbb{R}^n \times [0,T] \setminus (B(0,\frac{1}{\epsilon}) \times [\epsilon,T])} \phi(y,s)(\partial_t f - \Delta f)(x-y,t-s)dyds\Big| \leq \sup_{\mathbb{R}^n \times [0,T]} (|\partial_t f| + |\Delta f|) \iint\limits_{\mathbb{R}^n \times [0,T] \setminus (B(0,\frac{1}{\epsilon}) \times [\epsilon,T])} \phi$$

which $\to 0$ as $\epsilon \to 0$. Similarily:

$$\int\limits_{\mathbb{R}^n\times[0,T]\setminus(B(0,\frac{1}{\epsilon})\times[\epsilon,T])} \phi(y,t)f(x-y,0)dy \to 0 \text{ as } \epsilon \to 0$$

Also, the boundary term in the Laplacian integration by parts goes to zero as $\epsilon \to 0$. Hence we obtain that:

$$\partial_t u - \Delta u = f \text{ in } \mathbb{R}^n \times (0,T]$$

And the boundary conditions are satisfied, check lecture 11 for any more details. ☐

Theorem 3.14. (Uniqueness in $\mathbb{R}^n$ for Cauchy Problem)
There exists at most one solution $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C^0(\overline{\mathbb{R}^n \times (0,T]})$ of the problem:

$$\begin{cases} \partial_t u = \Delta u + f & \text{in } \mathbb{R}^n \times (0,T] \\ u(t,0) = h & \text{in } \mathbb{R}^n \end{cases}$$

which satisfies $|u(x,t)| \leq Ae^{\alpha|x|^2}(\star)\quad \forall (x,t) \in \mathbb{R}^n \times [0,T]$ for some constants $A, \alpha > 0$. However, there exist infinitely many solutions for which do not satisfy the growth condition of the problem with $f, h = 0$.

Proof. Uniqueness: By considering $u = u_1 - u_2$ we may assume that $f = h = 0$. For every $\epsilon > 0$ we define:

$$v_\epsilon(x,t) = u(x,t) - \frac{\epsilon}{(2T-t)^{n/2}}e^{\frac{|x|^2}{4(2T-t)}} \quad \forall (x,t) \in \mathbb{R}^n \times (0,T]$$

Then

$$\begin{aligned} \partial_\nu v_\epsilon(x,t) - \Delta v_\epsilon(x,t) &= \partial_\nu u(x,t) - \Delta u(x,t) - \frac{\epsilon}{(2T-t)^{n/2}}e^{\frac{|x|^2}{4(2T-t)}}\left(\frac{n}{2(2T-t)} + \frac{|x|^2}{4(2T-t)^2}\right) \\ &+ \frac{\epsilon}{(2T-t)^{n/2}}\text{div}\left(e^{\frac{|x|^2}{4(2T-t)}}\frac{x}{2(2T-t)}\right) \\ &= 0 \end{aligned}$$

And

$$v_\epsilon(x,t) = u(x,0) - \frac{\epsilon}{(2T)^{n/2}}e^{\frac{|x|^2}{8T}} < 0$$

and by $(\star)$,

$$x \in \partial B(0,R) \iff v_\epsilon(x,t) \leq Ae^{\alpha R^2} - \frac{\epsilon}{(2T)^{n/2}}e^{\frac{R^2}{8T}} < 0 \quad \text{for large } R$$

provided that $\frac{1}{8T} > \alpha$. By applying the weak maximum principle in $B(0,R) \times [0,T]$ it follows that $v_\epsilon \leq 0$ in $B(0,R) \times [0,T]$. By letting $R \to \infty$, we obtain that $v_\epsilon \leq 0$ in $\mathbb{R}^n \times [0,T]$. Finally, by letting $\epsilon \to 0$, we obtain that $u < 0$ in $\mathbb{R}^n \times [0,T]$. By considering $-u$ instead of u, the same argument gives $u \geq 0$ so $u \equiv 0$ in $\mathbb{R}^n \times [0,T]$.

If $T \geq \frac{1}{8\alpha}$ then apply the result successively to $u(x, T - \frac{k}{16\alpha})$ in $\mathbb{R}^n \times [0, \max(\frac{1}{16\alpha}, T - \frac{k}{16\alpha})]$.

Non-uniqueness (Tychonoff solutions): For every $\alpha \in (0,\infty)$ define $u_\alpha : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$, by:

$$u_\alpha(x,t) = \sum_{k=0}^{\infty} \frac{v_\alpha^\Delta(t)x_1^{2k}}{(2k)!} \quad \forall (x,t) \in \mathbb{R}^n \times (0,\infty)$$

and

$$v_\alpha(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ e^{-\frac{1}{\alpha t}}, & \text{if } t > 0 \end{cases}$$

The convergence and regularity of u_α are left as an exercise.

$$\partial_t u_\alpha(x,t) - \Delta u_\alpha(x,t) = \sum_{k=0}^{\infty} \frac{v_\alpha^{(k+1)}(t)x_1^{2k} - v_\alpha^{(k)}(t)2k(2k-1)x_1^{2k-2}}{(2k)!} \tag{8}$$

$$= \sum_{k=1}^{\infty} \frac{v_\alpha^{(k)}(t)x_1^{2(k-1)}}{2(k-1)!} - \sum_{k=1}^{\infty} \frac{v_\alpha^{(k)}(t)x_1^{2(k-1)}}{2(k-1)!} \tag{9}$$

$$= 0 \tag{10}$$

Moreover, $v_\alpha(x,t) \to 0$ at $t \to 0$ for all $x \in \mathbb{R}^n$. □

Theorem 3.15. (Regularity)
Any solution of the heat equation in U_T is smooth in U_T.

Proof. Let $(x_0, t_0) \in U_T$. Let $r > 0$ be small enough such that $3r < T$ and $B(x_0, 3r) \subset U$. Let $\zeta \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ be such that $\zeta = 0$ in $\mathbb{R}^{n+1} \setminus B((x_0,t_0), 2r)$ and $\zeta = 1$ in $B((x_0,t_0), r)$. Assume first that $t_0 < T$. Let $(\eta_\epsilon)_{\epsilon>0}$ be a family of mollifiers in $\mathbb{R}^{n+1}$. Define

$$u_\epsilon = \eta_\epsilon \star u \text{ and } v_\epsilon = \begin{cases} \zeta u_\epsilon & \text{in } B((x_0,t_0), r) \cap U_{t_0} \\ 0 & \text{in } U_{t_0} \setminus B((x_0,t_0), 2r) \end{cases}$$

It follows from previous existence and uniqueness theorems that:

$$\forall (x,t) \in B((x_0,t_0), r), v_\epsilon(x,t) = \int_0^r \int_{\mathbb{R}^n} \phi(x-y, t-s)(\partial_t v_\epsilon - \Delta v_\epsilon)(y,s) dyds$$

Observe that $\partial_t \zeta = \Delta \zeta = |\nabla \zeta| = 0$ in $\mathbb{R}^n \setminus A_0$ where $A_0 = U_{t_0} \cap \overline{B((x_0,t_0), 2r)} \setminus B((x_0,t_0), r)$ and $\forall (y,s) \in A_0$, $|(x-y, t-s)| \geq |(y-x_0, s-t_0)| - |(x-x_0, t-t_0)| \geq r - |(x-x_0, t-t_0)| > 0$. Since A_0 is compact and ϕ is continuous in $(\mathbb{R}^n \times [0,\infty)) \setminus \{(0,0)\}$, it follows from our previous calculations that we can pass to the limit as $\epsilon \to 0$ to obtain:

$$\forall (x,t) \in B((x_0,t_0), r), \quad v(x,t) = \int_0^t \int_{\mathbb{R}^n} \phi(x-y, t-s)(u)_\nu \zeta + u\Delta\zeta - 2 < \nabla\zeta, \Delta u > dyds$$

By continuity, this formula remains valid for $t = T$. By using all of the equations above, since ϕ is smooth in $(\mathbb{R}^n \times [0,\infty)) \setminus \{(0,0)\}$ it follows that u is smooth in $U_{t_0} \cap B((x_0,t_0), r)$ and in particular at (x_0, t_0). □

3.4 The Wave Equation

Definition 3.10. (The Wave Equation)
The wave equation is of the form:

$$\partial_t^2 u = \Delta u \text{ in } U \times I$$

For the case $n = 1$ we have:

$$\partial_t^2 u - \Delta u = \partial_t^2 u - \partial_x^2 u = (\partial_t - \partial_x)(\partial_t + \partial_x)u$$

Then by a change of variables $y = x + t, z = x - t$ we obtain the following:

$$\partial_x = \partial_y + \partial_z, \quad \partial_t = \partial_y - \partial_z, \quad \partial_x + \partial_t = 2\partial_y, \quad \partial_t - \partial_x = -2\partial_z$$

hence $\partial_t^2 u - \Delta u = -4\partial_y \partial_z = 0$, which gives that

$$u(x,t) = F(x+t) + G(x-t), \quad F, G \in C^2$$

Observations:

1. u is in general not better than C^2

2. There is no maximum principle or harnack's inequality. For example, if $F(s) = (s-1)^2$ and $G(s) = (s+1)^2$ then $u \geq 0$ in $\mathbb{R} \times (0, \infty)$ but

$$u(x,t) = 0 \iff \begin{cases} x+t-1 &= 0 \\ x-t+1 &= 0 \end{cases} \iff \begin{cases} x &= 0 \\ t &= 1 \end{cases}$$

What still works?

1. Energy methods

2. Duhamel's Principle

Theorem 3.16. (Uniqueness)
Assume that U is bounded and ∂U is C^1. Then there exists at most one solution $u \in C^2(\overline{U} \times [0,T])$ of the problem:

$$\begin{cases} \partial_t^2 u = \Delta u + f & \text{in } U \times (0,T] \\ u = g, \partial_t u = h & \text{on } \partial U \times (0,T] \\ u(\cdot,0) = h_0, & \text{in } U \\ \partial_t u(\cdot,0) = h_1 & \text{in } U \end{cases}$$

Proof. Assume that there are two solutions u_1 and u_2. Define $u = u_1 - u_2$. Then u is a solution of the problem:

$$\begin{cases} \partial_t^2 u = \Delta u & \text{in } U \times (0,T] \\ u = 0, \partial_t u = 0 & \text{on } \partial U \times (0,T] \\ u(\cdot,0) = 0, & \text{in } U \\ \partial_t u(\cdot,0) = 0 & \text{in } U \end{cases}$$

By the energy method, we define:

$$E(u)(t) = \int_U (|D_x u(\cdot,t)|^2 + (\partial_t u(\cdot,t))^2) dx$$

Then by taking the derivative with respect to t:

$$E'(u)(t) = 2\int_U (< D_x u(x,t), D_x \partial_t u(x,t) > + \partial_t u(x,t) \partial_t^2 u(x,t)) dx$$

We then proceed by integration by parts to obtain:

$$E'(u)(t) = 2\int_U \operatorname{div}(\partial_t u D_x u)(x,t) dx = 2\int_U \partial_t u(x,t) \partial_\nu u(x,t) ds(x) = 0$$

since $\forall (x,t) \in \partial U \times [0,T], u(x,t) = 0$, hence $\partial_t u(x,t) = 0$. Thus we obtain:

$$E(u)(t) = E(u)(0) = \int_U (|D_x u(x,0)|^2 + (\partial_t u(x,0))^2) dx = 0$$

since $u(\cdot,0) = \partial_t u(\cdot,0) = 0$ in U. It follows that $|D_x u| = \partial_t u = 0$ in $U \times [0,T]$ and since $u(\cdot,0) = 0$ on U, we obtain that $u = 0$ in $U \times [0,T]$. □

A specific phenomenon for wave-type equations:

Theorem 3.17. (Finite Propogation Speed)
Assume that $u \in C^2(\overline{U} \times [0,T])$ is a solution of the wave equation in $U \times [0,T]$ such that $u(\cdot,0) = \partial_t u(\cdot,0) = 0$ in $\overline{B(x_0,t_0)} \times \{t\} \subset U$. Then $u = 0$ in the cone $\bigcup_{t\in(0,T)} B(x_0, t_0 - t)$.

Proof. Define $E(u)(t) = \int_{B(x_0,t_0-t)}(|D_x u(x,t)|^2 + (\partial_t u(x,t))^2)dx$. Then we have that its derivative is:

$$E'(u)(t) = 2\int_{\partial B(x,t_0,t)} \partial_t u(x,t)\partial_\nu u(x,t)ds(x) - \int_{\partial B(x_0,t_0-t)}(|D_x u(x,t|^2 + (\partial_t u(x,t))^2)dx$$

Moreover, $2\partial_t u \partial_\nu u \leq 2|\partial_t u||D_x u| \leq |D_x u|^2 + (\partial_t u)^2$. Hence we obtain that $E'(u)(t) \leq 0$. Thus $E(u)(t) \leq E(u)(0) = 0$. On the other hand, $E(u)(t) \geq 0$ since $|D_x u|^2 + (\partial_t u)^2 \geq 0$. Thus we obtain $u = 0$ in the cone. ☐

Corollary 3.2. Let $u \in C^2(\mathbb{R}^n \times [0,T])$ be a solution of:

$$\begin{cases} \partial_t^2 u = \Delta u & \text{in } \mathbb{R}^n \times (0,T] \\ u(\cdot,0) = h_0 & \text{in } \mathbb{R}^n \\ \partial_t u(\cdot,0) = h_1 & \text{in } \mathbb{R}^n \end{cases}$$

If h_0 and h_1 have compact support in $\overline{B(x_0,r_0)}$ then for each $t \in (0,T]$, $u(\cdot,t)$ has compact support in $\overline{B(x_0,r_0+t)}$.

Proof. Apply the previous theorem to the cones $\bigcup_{t\in(0,|x-x_0|-r_0)} B(x, |x-x_0| - r_0 - t) \times \{t\}$ for all $x \in \mathbb{R}^n \setminus \overline{B(x_0,r_0)}$ ☐

Theorem 3.18. (Duhamel's Principle)
Let $f : U \times (0,T] \to \mathbb{R}$, $g : \partial U \times (0,T] \to \mathbb{R}$, $h_0, h_1 : U \to \mathbb{R}$, $v_0 \in C^2(U_T)$ be a solution of:

$$\begin{cases} \partial_t^2 v_0 = \Delta u & \text{in } U \times (0,T] \\ v_0 = g, \partial_t v_0 = h & \text{on } \partial U \times (0,T] \\ v_0(\cdot,0) = h_0 & \text{in } U \\ \partial_t v_0(\cdot,0) = h_1 & \text{in } U \end{cases}$$

and $v \in C^0(\{(x,t-s,s) : x \in \overline{U}, t \in (0,T], s \in [0,t]\})$ is such that for each $s \in (0,T)$, $v_s = v(\cdot,\cdot,s) \in C^2(U_{T-s})$ is a solution of:

$$\begin{cases} \partial_t^2 v_s = \Delta v_s & \text{in } U \times (0,T-s] \\ v_s = 0, \partial_t v_s = 0 & \text{on } \partial U \times (0,T-s] \\ v_s(\cdot,0) = 0 & \text{in } U \partial_t v_s(\cdot,0) = f(\cdot,s) \quad \text{in } U \end{cases}$$

Assume moreover that for each compact subset K of $\{(x,t-s,s) : x \in U, 0 < t \leq T, 0 \leq s \leq t\}$, $\partial_t v_s(x,t), D_x v_s(x,t), D_x^2 v_s(x,t)$ are uniformly continuous in $K \cap \bigcup_{0<s<T}(U_{T-s})$. Then the function $u : U \times (0,T) \to \mathbb{R}$ defined as $u(x,t) = v_0(x,t) + \int_0^t v_s(x,t-s)ds$ is a solution of:

$$\begin{cases} \partial_t^2 u = \Delta u + f & \text{in } U \times (0,T] \\ u = g, \partial_t u = h & \text{on } \partial U \times (0,T] \\ u(\cdot,0) = h_0 & \text{in } U \\ \partial_t u(\cdot,0) = h_1 & \text{in } U \end{cases}$$

Proof. The regularity of u follows from the given assumptions. We verify it is a solution to the equation:

$$\begin{aligned}\partial_t^2 u(x,t) &= \partial_t^2 v_0(x,t) + \frac{d}{dt}[v_t(x,0) = \int_0^t \partial_t v_s(x,t-s)ds] \\ &= \partial_t^2 v_0(x,t) + \partial_t v_t(x,0) + \int_0^t \partial_t^2 v_s(x,t-s)ds \\ &= \Delta u(x,t) + f(x,t)\end{aligned}$$

We now verify that the boundary conditions are satisfied:

$$\forall x \in \partial U \quad u(x,t) = v_0(x,t) + \int_0^t v_s(x,t-s)ds = g(x)$$

$$\forall x \in U \begin{cases} u(x,0) = v_0(x,0) = h_0(x) \\ \partial_t u(x,0) = \partial_t v_0(x,0) + v_t(x,0) = h_1(x) \end{cases}$$

□

We will now consider the case where $U = \mathbb{R}^n, T = \infty, f = 0$. We will consider the problem:

$$\begin{cases} \partial_t^2 u = \Delta u & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(\cdot,0) = h_0 & \text{in } \mathbb{R}^n \\ \partial_t u(\cdot,0) = h_1 & \text{in } \mathbb{R}^n \end{cases}$$

Case n=1: In this case we obtain that $u(x,t) = F(x+t) + G(x-t)$, where $F, G \in C^2(\mathbb{R})$. We verify:

$$\begin{cases} F(x) + G(x) &= h_0(x) \\ F'(x) - G'(x) &= h_1(x) \end{cases} \iff \begin{cases} F(x) + G(x) &= h_0(x) \\ F(x) - G(x) &= \int_0^x h_1(s)ds + F(0) - G(0) \end{cases} \iff \begin{cases} F(x) &= \frac{1}{2}(h_0(x) + \int_0^x h_1(s)ds \\ G(x) &= \frac{1}{2}(h_0(x) - \int_0^x h_1(s)ds \end{cases}$$

This gives D'Alembert's Formula:

$$u(x,t) = \frac{1}{2}(h_0(x+t) + h_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h_1(s)ds$$

Case $n \geq 2$: We will use the method of spherical means. For each $x \in \mathbb{R}^n, r > 0, t \geq 0$, define $U_x(r,t) = ⨍_{\partial B(x,r)} u(y,t)dS(y)$ and $H_{i,x}(r,t) = ⨍_{\partial B(x,r)} h_i(y,t)dS(y), \quad i \in \{0,1\}$. Then we have that:

$$\partial_r U_x(r,t) = \frac{1}{\int_{\partial B(x,r)} 1} \int_{B(x,r)} \Delta u(y,t)dy$$

Recall $\int_{\partial B(x,r)} 1 = r^{n-1} \int_{\partial B(0,1) 1}$. We then obtain:

$$\partial_r U_x(r,t) = \frac{1}{r^{n-1}} \int_{B(x,r)} \Delta u(y,t)dy$$

For the second derivative we obtain:

$$\partial_r^2 U_x(r,t) = ⨍_{\partial B(x,r)} \partial_t^2 u(y,t)dS(y) - \frac{(n-1)}{r\int_{\partial B(x,r)} 1} \int_{B(x,r)} \partial_t^2 u(y,t)dy$$

We have that the first integral on the right hand side is equal to $\partial_t^2 U_x(r,t)$ and the second integral is equal to $\frac{-(n-1)}{r}\partial_r U_x(r,t)$. Thus we obtain that:

$$\partial_r^2 U_x + \frac{(n-1)}{r}\partial_r U_x = \partial_t^2 U_x$$

Therefore U_x solves:

$$\begin{cases} \partial_t^2 U_x = \partial_r^2 U_x + \frac{(n-1)}{r}\partial_r U_x & \text{in } (0,\infty)\times(0,\infty) \\ U_x(\cdot,0) = H_{0,x}(r,0) & \text{in } (0,\infty) \\ \partial_t U_x(\cdot,0) = H_{1,x}(r,0) & \text{in } (0,\infty) \end{cases}$$

Case n=3: Define $\tilde{U_x}(r,t) = rU_x(r,t)$. Then $\partial_r^2 \tilde{U_x} = \partial_r(U_x + r\partial_r U_x) = 2\partial_r U_x + r\partial_r^2 U_x = r\partial_r^2 U_x = r\partial_t^2 \tilde{U_x}$. And $\tilde{U_x}(0,t) = 0$, which gives:

$$\begin{cases} \partial_t^2 \tilde{U_x} = \partial_r^2 \tilde{U_x} & \in (0,\infty)^2 \\ \tilde{U_x}(0,t) = 0 & \text{in } (0,\infty) \\ \tilde{U_x}(\cdot,0) = rH_{0,x}(r,0) & \text{in } (0,\infty) \\ \partial_t \tilde{U_x}(\cdot,0) = rH_{1,x}(r,0) & \text{in } (0,\infty) \end{cases}$$

We obtain $\tilde{U_x}(r,t) = F(r+t) + G(r-t), \quad F, G \in C^2(\mathbb{R})$. Then we have:

$$\begin{cases} F(r) + G(r) = rH_{0,x}(r) & \forall r > 0 \\ F'(r) - G'(r) = rH_{1,x}(r) & \forall r > 0 \\ F(t) + G(-t) = 0 & \forall t > 0 \end{cases} \iff \begin{cases} F(r) = \frac{1}{2}(rH_{0,x}(r) + \int_0^r sh_{1,x}(s)ds) \\ G(r) = \frac{1}{2}(rH_{0,x}(r) - \int_0^r sh_{1,x}(s)ds) \end{cases}$$

which gives the formula:

$$\tilde{U_x}(x,t) = \frac{1}{2}((r+t)H_{0,x}(r+t) + (r-t)H_{0,x}(t-r) + \int_{t-r}^{t+r} sH_{1,x}(s)ds)$$

provided that $r - t < 0$. The case $r > t$ is useless since we let $r \to 0$. Then $u(x,t) = \lim_{r\to 0} U_x(r,t) = \lim_{r\to 0} \frac{1}{r}\tilde{U_x}(r,t)$. We obtain that:

$$= \lim_{r\to 0}\left(\frac{(r+t)H_{0,x}(r+t) - (t-r)H_{0,x}(t-r)}{2r} + \fint_{t-r}^{t+r} sH_{1,x}(s)ds\right)$$

$$= \frac{d}{dt}[tH_{0,x}(t)] + tH_{1,x}(t) = \frac{d}{dt}[t\fint_{\partial B(x,t)} h_0] + t\fint_{\partial B(x,t)} h_1$$

provided that $h_0 \in C^3(\mathbb{R}^3)$ and $h_1 \in C^2(\mathbb{R}^3)$. This is Kirchoff's Formula.

Case n=2: We define $\overline{u}(x_1,x_2,x_3,t) = u(x_1,x_2,t)$ (fixing a variable). $\overline{h_i}(x_1,x_2,x_3) = h_i(x_1,x_2), \quad i \in \{0,1\}$ so that $\overline{u}$ solves:

$$\begin{cases} \partial_t^2 \overline{u} = \Delta\overline{u} & \text{in } \mathbb{R}^3\times(0,\infty) \\ \overline{u}(\cdot,0) = \overline{h_0} & \text{in } \mathbb{R}^3 \\ \partial_t \overline{u}(\cdot,0) = \overline{h_1} & \text{in } \mathbb{R}^3 \end{cases}$$

Then Kirchoff's Formula gives:

$$u(x,t) = \frac{d}{dt}\left[t\fint_{\partial B(\overline{x},t)} \overline{h_0}\right] + t\fint_{\partial B(\overline{x},t)} \overline{h_1}$$

where $\overline{x} = (x_1, x_2, x_3 = 0)$ since x_3 can be chosen arbitrarily. We use the parametrization:

$$\psi_\pm : B(x,t) \subset \mathbb{R}^2 \to \partial_\pm B(x,t)$$

representing the upper and lower hemisphere of $\partial B(x,t) \subset \mathbb{R}^2$ such that $y \mapsto (y, \pm\sqrt{t^2 - |x-y|^2})$. For each $i \in \{0,1\}$ we have that: $t\fint_{\partial B(x,t)} \overline{h_i} = t\fint_{\partial B(x,t)} h_i(y)dS(y) = \frac{2t}{4\pi t^2}\int_{B(x,t)} h_i(y)|\partial_{y_1}\psi_\pm \wedge \partial_{y_2}\psi_\pm|dy$ Then we have that:

$$\partial_{y_1}\psi_\pm = (1, 0, \frac{\pm x_1 - y_1}{\sqrt{t^2 - |x-y|^2}}), \quad \partial_{y_2}\psi_\pm = (0, 1, \frac{\pm x_2 - y_2}{\sqrt{t^2 - |x-y|^2}})$$

Thus we have that:

$$|\partial_{y_1}\psi_\pm \wedge \partial_{y_2}\psi_\pm|^2 = |(\frac{\pm y_1 - x_1}{\sqrt{t^2-|x-y|^2}}, \frac{\pm y_2 - x_2}{\sqrt{t^2-|x-y|^2}}, 1)|^2 = \frac{|x-y|^2}{t^2-|x-y|^2} + 1 = \frac{t^2}{t^2-|x-y|^2}$$

Which gives:

$$t \fint_{\partial B(x,t)} \overline{h_i} = \frac{1}{2\pi t}\int_{B(x,t)} h_i(y)\frac{dy}{\sqrt{t^2-|x-y|^2}} = \frac{t}{2\pi}\int_{B(0,1)} h_i(x+t,z)\frac{dz}{\sqrt{1-z^2}}$$

Moreover,

$$\begin{aligned}\frac{d}{dt}\left[t\fint_{\partial B(x,t)} \overline{h_0}\right] &= \frac{1}{2\pi}\int_{B(0,1)} (h_0(x+t,r) + < Dh_0(x+tz), z >)\frac{dz}{\sqrt{1-|z|^2}} \\ &= \frac{1}{2\pi t}\int_{B(x,t)} (h_0(y) + < Dh_0(y), y-x >)\frac{dz}{\sqrt{t^2-|x-y|^2}}\end{aligned}$$

Finally, we obtain Poisson's Formula:

$$u(x,t) = \frac{t}{2}\fint_{B(x,t)} (th_1(y) + h_0(y) + t < Dh_0(y), y-x >)\frac{dz}{\sqrt{t^2-|x-y|^2}}$$

provided that $h_0 \in C^3(\mathbb{R}^2)$ and $h_1 \in C^2(\mathbb{R}^2)$.

Case $n \geq 4$: If n is odd, we define: $\tilde{U_x}(r,t) = (\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k+1}U_x(r,t))$ and $\tilde{H_{i,x}}(r,t)$ for $i \in \{0,1\}$ likewise. Then we have that $\tilde{U_x}$ solves:

$$\begin{cases} \partial_t^2 \tilde{U_x} = \partial_r^2 \tilde{U_x} & \text{in } (0,\infty)\times(0,\infty) \\ \tilde{U_x}(0,\cdot) = 0 & \text{in } (0,\infty) \\ \tilde{U_x}(\cdot,0) = \tilde{H_{0,x}}(r,0) & \text{in } (0,\infty) \\ \partial_t\tilde{U_x}(\cdot,0) = \tilde{H_{1,x}}(r,0) & \text{in } (0,\infty) \end{cases}$$

After integrating $k-1$ times to obtain a formula for U_x, we let $r \to 0$ to obtain a formula for u:

$$u(x,t) = \frac{1}{1\times 3\times\ldots\times(2k-1)}\left(\frac{d}{dt}\left[(\frac{1}{t}\frac{d}{dt})^{k-1}(t^{2k-1}\int_{\partial B(x,t)} h_0)\right] + (\frac{1}{t}\frac{d}{dt})^{k-1}(t^{2k-1}\fint_{\partial B(x,t)} h_1)\right)$$

provided that $h_0 \in C^{k+2}(\mathbb{R}^n)$ and $h_1 \in C^{k+1}(\mathbb{R}^n)$.

If n is even, we define $\overline{U}(x_1, x_2, \ldots, x_{2k}, t) = u(x_1, x_2, \ldots, x_{2k}, t)$ and $\overline{h_i}$ likewise so that $\overline{U}$ solves:

$$\begin{cases} \partial_t^2\overline{U} = \Delta\overline{U} & \text{in } \mathbb{R}^n\times(0,\infty) \\ \overline{U}(\cdot,0) = \overline{h_0} & \text{in } \mathbb{R}^n \\ \partial_t\overline{U}(\cdot,0) = \overline{h_1} & \text{in } \mathbb{R}^n \end{cases}$$

Use the formula in $\mathbb{R}^{2k+1}$ to obtain a formula for u and then the change of variables $\psi_\pm$ to rewrite it as:

$$\begin{aligned}u(x,t) = &\frac{1}{1\times 2\times\ldots\times 2k}\frac{d}{dt}\left[(\frac{1}{t}\frac{d}{dt})^{k-1}(t^{2k}\fint_{B(x,r)} h_0(y)\frac{dy}{\sqrt{t^2-|x-y|^2}})\right] \\ &+ \frac{1}{1\times 2\times\ldots\times 2k}(\frac{1}{t}\frac{d}{dt})^{k-1}(t^{2k}\fint_{B(x,r)} h_1(y)\frac{dy}{\sqrt{t^2-|x-y|^2}})\end{aligned}$$

provided that $h_0 \in C^{k+2}(\mathbb{R}^n)$ and $h_1 \in C^{k+1}(\mathbb{R}^n)$.

4 The Modern Approach Based on Integration

4.1 Introduction

Quasilinear PDEs can sometimes be rewritten using integration.

Example 4.1. $\partial_t u + \text{div}_x(F(u)) = f$ in $U \times (0,T)$ where f is given and $F : \mathbb{R} \to \mathbb{R}^n$.
$\forall v \in C_c^\infty(U \times (0,T))$, we have that:

$$-\int_0^T \int_U v(\partial_t u + \text{div}_x(F(u))) = \int_0^T \int_U (u\partial_t v + < F(u), D_x v >) = -\int_0^T \int_U fv$$

Example 4.2. $\partial_t u = \Delta u + f$ in $U \times (0,T)$
$\forall v \in C_c^\infty(U \times (0,T))$ we have that:

$$\int_0^T \int_U v(\partial_t u - \Delta u) = \int_0^T \int_U u\partial_t v - < Du, Dv > = \int_0^T \int_U u(\partial_t v + \Delta v) = \int_0^T \int_U fv$$

Example 4.3. $\Delta_p u = f$ in U where $\Delta_p u = \text{div}(|Du|^{p-2} Du)$
$\forall v \in C_c^\infty(U)$ we have that:

$$\int_U v\Delta_p u = -\int_U |Du|^{p-2} < Du, Dv > = \int_U fv$$

Key Observation: Do not need $u \in C^1$ or $u \in C^2$ for these integrals to exist.
Idea: We want to use integration to define a weak-type of solution, then prove the existence of these solutions:

(i) either in addition to classical solutions

(ii) or in situations where classical solutions do not exist

(iii) or as a tool to then show that these solutions are classical solutions

Now we will go over key points that we need to know about integration and L^p spaces.

Definition 4.1. Let U be an open subset of $\mathbb{R}^n$ or more generall, measurable subsets of $\mathbb{R}^n$. For each $p \in [1, \infty]$ we define:

$$L^p(U) = \begin{cases} \{u : U \to \overline{\mathbb{R}} \text{ measurable such that } \int |u|^p < \infty\} \text{ if } p \in [1,\infty) \\ \{u : U \to \overline{\mathbb{R}} \text{ measurable such that } \exists c > 0 \text{ such that } |u| \leq c \text{ a.e. in } U\} \text{ if } p = \infty \end{cases}$$

We identify functions in $L^p(U)$ which are equal a.e. Moreover, we define:

Definition 4.2.

$$||u||_{L^p(U)} = \begin{cases} (\int_U |u|^p)^{\frac{1}{p}} \text{ if } p \in [1,\infty) \\ \inf\{c > 0 : |u| \leq c \text{ a.e. in } U\} \text{ if } p = \infty \end{cases}$$

Proposition 4.1. $|u| \leq ||u||_{L^\infty(U)}$ a.e. in U.

Theorem 4.1. (Holder's Inequality) Define $p' = \begin{cases} \frac{p}{p-1} \text{ if } p \in (1,\infty) \\ 1 \text{ if } p = \infty \infty \text{ if } p = 1 \end{cases}$ Then for every $u \in L^p(U)$ and $v \in L^{p'}(U)$ we have that $uv \in L^1(U)$ and:

$$||uv||_{L^1(U)} \leq ||u||_{L^p(U)} ||v||_{L^{p'}(U)}$$

Theorem 4.2. (Minkowski's Inequality) For every $u, v \in L^p(U)$ we have that:

$$||u+v||_{L^p(U)} \leq ||u||_{L^p(U)} + ||v||_{L^p(U)}$$

Theorem 4.3. (Riesz-Fischer) $(L^p(U), ||\cdot||_{L^p(U)})$ is a complete normed vector space (Banach space). Moreover, for every sequence $(u_k)_{k\in\mathbb{N}}$ in $L^p(U)$ such that $u_k \to u$ in $L^p(U)$, there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ such that $u_{k_j} \to u$ a.e. in U.

Proposition 4.2. (Hilbert Space) Define $< u, v >_{L^2(U)} = \int_U uv \quad \forall u, v \in L^2(U)$. $(L^2(U), < \cdot, \cdot >_{L^2(U)})$ is a Hilbert space, i.e. $< \cdot, \cdot >_{L^2(U)}$ is an inner product and moreover $(L^2(U), ||\cdot||_{L^2(U)})$ is a Banach space, with $||u||_{L^2(U)} = \sqrt{< u, v >_{L^2(U)}}$.

Proposition 4.3. If U is bounded (or more generally has finite measure) and $1 \leq p_1 \leq p_2 \leq \infty$ then $\exists c > 0$ such that $\forall u \in L^{p_2}(U) \quad ||u||_{L^{p_1}} \leq c||u||_{L^{p_2}(U)}$. In particular, $L^{p_2}(U) \subseteq L^{p_1}(U)$.

Proof. If $p_1 < p_2 = \infty$ then $\left(\int_U |u|^{p_1}\right)^{\frac{1}{p_1}} \leq \left(||u||_\infty^{p_1} \int_U 1\right)^{\frac{1}{p_1}} = ||1||_{L^2(U)}^{\frac{1}{p_1}} ||u||_\infty$. If $p_1 < p_2 < \infty$ then $\left(\int_U |u|^{p_1}\right)^{\frac{1}{p_1}} \leq \left(\int_U |u|^{p_2}\right)^{\frac{p_1}{p_2}} \left(\int_U (1)^{(\frac{p_2}{p_1})'}\right)^{\frac{1}{(\frac{p_2}{p_1})'}} = ||1||_{L^1(U)}^{\frac{p_2-p_1}{p_1 p_2}} ||u||_{L^{p_2}(U)} < \infty$ since U is bounded. $\square$

Theorem 4.4. (Fubini-Tonelli) Let $U_1 \subset \mathbb{R}^{n_1}, U_2 \subset \mathbb{R}^{n_2}$ be open sets. Then:

$$\iint_{U_1 \times U_2} u(x,y)dxdy = \int_{U_1} \left(\int_{U_2} u(x,y)dy \right) dx = \int_{U_2} \left(\int_{U_1} u(x,y)dx \right) dy$$

provided that f is non-negative and measurable in $U_1 \times U_2$ (Tonelli) or $f \in L^1(U_1 \times U_2)$ (Fubini).

Theorem 4.5. (Dominated Convergence Theorem) Assume that $1 \leq p < \infty$. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $L^p(U)$ such that $u_k \to u$ a.e. in U and $\exists v \in L^p(U)$ such that $|u_k| \leq v$ a.e. in U for all $k \in \mathbb{N}$. Then $u \in L^p(U)$ and $u_k \to u$ in $L^p(U)$.

Theorem 4.6. (Monotone Convergence Theorem) Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $L^1(U)$ such that $u_k \geq 0$ a.e. in U and $0 \leq u_k(x) \leq u_{k+1}(x) \quad \forall k \in \mathbb{N}, x \in U$. Then u is measureable, non-negative a.e. in U and $\int_U u = \lim_{k\to\infty} \int_U u_k$.

These convergence results yield the following:

Proposition 4.4. Assume that $1 \leq p < \infty$. Then the following subspaces are dense in $(L^p(U), ||\cdot||_{L^p(U)})$:

1. Step functions (sums of characteristic functions on rectangles)
2. $C_c^0(U)$ continuous functions with compact support in U

4.2 Conservation Laws

We are considering the following problem:

$$\partial_t u + \mathrm{div}_x(F(u)) = 0 \quad \text{ in } U \times (0, T)$$

where $F : \mathbb{R} \to \mathbb{R}^n$.

Why do we consider the conservation law?

If $F \in C^1(\mathbb{R})$, $u \in C_c^1(\mathbb{R}^n \times [0, \infty))$ and $\forall t \in [0, \infty)$, assume $u(\cdot, t)$ and $F(u(\cdot, t))$ have compact support in $\mathbb{R}^n$ (the latter condition can be replaced by decay conditions on u and $F(u)$ as $|x| \to \infty$). Then we have that:

$$\frac{d}{dt}\left[\int_{\mathbb{R}} u dx\right] = \int_{\mathbb{R}} \partial_t u = -\int_{\mathbb{R}^n} \text{div}(F(u)) = 0$$

That is, $\int_{\mathbb{R}^n} u$ is constant in t.

We want to find a new concept of solutions to the following problem:

$$\begin{cases} \partial_t u + \text{div}_x F(u) = 0 & \text{in } \mathbb{R}^n \times (0, T) \quad (\star) \\ u(\cdot, 0) = h & \text{in } \mathbb{R}^n \end{cases}$$

If $u \in C^1(\mathbb{R}^n \times [0, T))$ and is a classical solution of $(\star)$, then $\forall v \in C_c^\infty(\mathbb{R}^n \times [0, T))$:

$$0 = -\int_0^T \int_{\mathbb{R}^n} v(\partial_t u + \text{div}_x(F(u))) = \int_0^T \int_{\mathbb{R}^n} (u\partial_t v + < F(u), D_x v >) + \int_{\mathbb{R}^n} u(x, 0)v(x, 0)dx$$

Definition 4.3. (Locally integrable functions) Define $L^p_{\text{loc}}(E) = \{u : E \to \overline{\mathbb{R}}$ measurable such that $\forall K \subset E$ compact $u|_K \in L^p(K)\}$.

Definition 4.4. (Integral solution/distribution solution) If $h \in L^1_{\text{loc}}(\mathbb{R}^n)$, $F \in C^0(\mathbb{R})$, and V be an open subset of $\mathbb{R}^n \times [0, T)$. We say that $u \in L^\infty_{\text{loc}}(V)$ is an integral solution (or a distribution solution) of the problem:

$$\begin{cases} \partial_t u + \text{div}_x F(u) = 0 & \text{in } V \cap (\mathbb{R}_n \times (0, T)) \quad (\star_V) \\ u(\cdot, 0) = h & \text{in } V \cap \mathbb{R}^n \times \{0\} \end{cases}$$

If $\forall v \in C_c^\infty(V)$ we have that:

$$\int_0^T \int_{\mathbb{R}^n} u\partial_t v + < F(u), D_x v > + \int_{\mathbb{R}^n} h(x)v(x, 0)dx = 0$$

Remark 4.1. The condition $u \in L^\infty_{\text{loc}}$ can be relaxed to $u \in L^1_{\text{loc}}$ and $F(u) \in L^1_{\text{loc}}$.

Proposition 4.5. If $F \in C^1(\mathbb{R})$ and $h \in C^0(\mathbb{R}^n)$ and $u \in C^1(V)$ then u is a classical solution $\iff$ u is an integral solution of $(\star_V)$.

Proof. ($\implies$) This was proven above. ($\impliedby$) $\forall v \in C_c^\infty(V)$ we have that:

$$\begin{aligned} \iint_V v(\partial_t u + \text{div}_x(F(U))) &= -\iint_V u(\partial_t v) + < F(u), D_x u > \\ &\quad - \int_{\{x:(x,0)\in V\}} u(x, 0)v(x, 0)dx \\ &= \int_{\{x:(x,0)\in V\}} (h(x) - u(x, 0))v(x, 0)dx \end{aligned}$$

Let $(\eta_\epsilon)_\epsilon$ be a family of mollifiers. Then by letting $v(y, s) = \eta_\epsilon(x-y, t-s)$ we obtain: $\forall (x, t) \in \mathbb{R}^n \times (0, \infty)$, $\eta_\epsilon \star (\partial_t u + \text{div}_x(F(u))) = 0$ for $\epsilon > 0$ small enough. Then for $t = 0$ we obtain that: $\forall v \in C_c^\infty(V)$, $\int_{\{x:(x,0)\in V\}} (h(x) - u(x, 0))v(x, 0) = 0$ which implies that $h(x) = u(x, 0)$ a.e. in $\mathbb{R}^n$ using mollifiers. $\square$

At points where u and F are C^1, the method of characteristics gives:

$$\begin{cases} x' = F'(z) \\ t' = 1 \\ z' = 0 \end{cases} \implies \begin{cases} x = F'(z_0)s + x_0 \\ t = s + t_0 \\ z = z_0 \end{cases}$$

Example 4.4. $\begin{cases} \partial_t u + \frac{1}{2}\partial_x(u^2) = 0 \\ u(x_0) = \begin{cases} c_0, & \text{if } x \leq 0 \\ c_0 + (c_1 - c_0), & \text{if } 0 \leq x \leq 1 \\ c_1, & \text{if } x \geq 1 \end{cases} \end{cases}$

Then we have that $F(z) = \frac{z^2}{z}$ and $F'(z) = z$. So $u(z_0 s + x_0, s + t_0) = z_0$. Letting $t_0 = 0$ and $z_0 = h(x_0)$ we obtain that $u(h(x_0)s + x_0, s) = h(x_0)$. Then:

$$\begin{cases} x = (c_0 + (c_1 - c_0))s + x_0 & \text{if } x_0 \leq 0 \\ t = s \end{cases} \implies \begin{cases} x_0 = \frac{x - c_0 t}{1 + (c_1 - c_0)t} \\ s = t \end{cases}$$

Thus we obtain that:

$$u(x,t) = \begin{cases} c_0 \text{ on the lines } s \mapsto (c_0 s + x_0, s) \text{ with } x_0 \leq 0 \text{ as long as } u \text{ is } C^1 \\ (c_0 + (c_1 - c_0)(\frac{x - c_0 t}{1 - (c_1 - c_0)t})) = \frac{c_0 t (c_1 - c_0) x}{1 + (c_1 - c_0)t} \text{ for } 0 < x_0 < 1 \text{ as long as } u \text{ is } C^1 \\ c_1 \text{ on the lines } s \mapsto (c_1 s + x_0, s) \text{ with } x_0 \geq 1 \text{ as long as } u \text{ is } C^1 \end{cases}$$

Consider $u(x,t) = \begin{cases} c_0 \text{ if } x < \min(c_0 t, c_1 t + 1) \text{ or } c_1 t + 1 < x < \overline{x}(t) \\ \frac{c_0 + (c_1 - c_0)x}{1 + (c_1 - c_0)t} \text{ if } c_0 t < x < c_1 t + 1 \\ c_1 \text{ if } x > \max(c_0 t, c_1 t + 1) \text{ or } \overline{x}(t) < x < c_0 t \end{cases}$ for some C^1 curve $\overline{x}(t)$, which is defined only in the case where $c_0 > c$ and when $t > \frac{1}{c_0 - c_1}$. That is, $c_1 t + 1 < c_0 t$.

Question: For which curve $\overline{x}(t)$ is u an integral solution?

More generally, we obtain:

Proposition 4.6. Let V and W be two open subsets of $\mathbb{R}^n \times (0, \infty)$ be such that $\partial V \in C^1$ and $W \cap \partial V \neq \emptyset$. Assume that $u \in C^1(\overline{V \cap W})$ and $u \in C^1(\overline{W} \setminus V)$ and u is a classical solution of:

$$\partial_t u + \operatorname{div}(F(U)) = 0 \quad \text{in } V \cap W \text{ and } W \setminus V$$

Then u is an integral solution of $\partial_t u + \operatorname{div}(F(U)) = 0$ in $W \iff (u_1 - u_0)\nu_t + < F(u_1) - F(u_0), \nu_x >= 0$ on $W \cap \partial V$. This is known as the Rankine-Hugomoit's condition. where $\nu = (\nu_x, \nu_t)$ is the outward normal vector on ∂V, u_0 and u_1 are the limits of $u(x,t)$ as $(x,t) \to \partial V$ on the side of V and $W \setminus V$ respectively.

Proof. $\forall v \in C_c^\infty(U)$, by applying the Divergence Theorem in $V \cap W$ and $W \setminus \overline{V}$, we get:

$$\begin{aligned} \iint_W u \partial_t v + < F(u), D_x v > &= -\iint_W v(\partial_t u + \operatorname{div}_x(F(u))) + \int_{W \cap \partial V} v(u_0 \nu_t + < F(u_0), \nu_x >) \\ &- \int_{W \cap \partial V} v(u_1 \nu_t + < F(u_1), \nu_x >) \\ &= \int_{W \cap \partial V} v(u_0 - u_1)\nu_t + < F(u_0) - F(u_1), \nu_x > \end{aligned}$$

It follows that u is an integral solution in $W \iff \forall v \in C_c^\infty(W)$,

$$\int_{W \cap \partial V} v(u_0 - u_1)\nu_t + < F(u_0) - F(u_1), \nu_x >$$

using straightening coordinates together with mollifiers we obtain that this is equivalent to Rankine-Hugomoit's condition. ☐

For our example, $W \cap \partial V = \{(\overline{x}(t), t) : t > \frac{1}{c_0 - c_1}\}$. Then we have that: $\nu = \frac{(1, \overline{x}'(t))}{\sqrt{(1)^2 + (-\overline{x}'(t))^2}}$ and we have that $u_0 = c_0$ and $u_1 = c_1$, $F(u_0) = \frac{c_0^2}{2}$ and $F(u_1) = \frac{c_1^2}{2}$. Hence the Rankine-Hugomoit's condition gives: $(c_1 - c_0)\overline{x}'(t) = \frac{c_1^2}{2} - \frac{c_0^2}{2}$ which gives $\overline{x}'(t) = \frac{c_1 - c_0}{2}$. By integrating we obtain:

$$\overline{x}(t) = \frac{(c_1 - c_0)t + 1}{2}$$

Exercise 4.1. Verify that u is an integral solution to the problem in $\mathbb{R}^n \times (0, \infty)$.

The concept of distribution solutions can be extended. For instance,

Definition 4.5. Let U be an open subset of $\mathbb{R}^n$, A a finite family of multi-indices, $a_\alpha \in L^1_{loc}(U)$ for each $\alpha \in A$ and $f \in L^1_{loc}(U)$. We call the distribution solution of:

$$\sum_{\alpha \in A} D^\alpha (a_\alpha u) = f \quad \text{ in } U$$

a function $u \in L^1_{loc}(U)$ such that $\forall v \in C_c^\infty(U)$ we have that:

$$\sum_{\alpha \in A} (-1)^{|\alpha|} \int_U a_\alpha u D^\alpha v = \int_U fv$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Proposition 4.7. If $a_\alpha \in C^{|\alpha|}(U)$, $u \in \bigcap_{\alpha \in A} C^{|\alpha|}(U)$, and $f \in C^0(U)$ then u is a classical solution $\iff$ u is an integral solution.

This concept can even be extended to cases where f is not a function.

Example 4.5. The Green's function of $-\Delta$ in U is said to be such that for each $x \in U$, $G_x(\cdot) = G(x, \cdot)$ for fixed x is a solution of $-\Delta G_x = \delta_x$ because $\forall v \in C_c^\infty(U)$, we have that $-\int_U G_x \Delta v = v(x)$.

4.3 Weak Derivatives and Sobolev Spaces

Recall that if u is a classical solution of $-\Delta u = f$ in U, then $\forall v \in C_c^\infty(U)$ we have:

$$-\int_U v \Delta u = \int_U < Du, Dv > = -\int_U u \Delta v = \int_U fv$$

More generally, for $p > 1$, if u is a classical solution of $-\Delta_p u = f$ in U then $\forall v \in C_c^\infty(U)$ we have:

$$-\int_U v \Delta_p u = \int_U |Du|^{p-2} < Du, Dv > = -\int_U u \Delta_p v = \int_U fv$$

We want to use $\int_U < Du, Dv >$ or $\int_U |Du|^{p-2} < Du, Dv >$ to define our solutions as it provides a better framework for existence. However, Du must exist in some weak sense, and belong to $L^p(U)$ for some p.

Definition 4.6. (Weak derivative) Let U be an open subset of $\mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ be a multi-index and $u \in L^1_{loc}(U)$. Then $w \in L^1_{loc}(U)$ is said to be the α-th weak derivative of u if $\forall v \in C_c^\infty(U)$,

$$\int_U u D^\alpha v = (-1)^{|\alpha|} \int_U wv$$

We denote $w = D^\alpha u$.

Example 4.6. Let $n = 1$. $u(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } x > 0 \end{cases}$ Then $u'(x) = w(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$ Then $u, w \in L^\infty_{loc} \subset L^1_{loc}$ and $\forall v \in C_c^\infty(\mathbb{R})$ we have:

$$\int_{\mathbb{R}} uv' = \int_0^\infty xv'(x) = [xv(x)]_0^\infty - \int_0^\infty v(x)dx = -\int_{\mathbb{R}} wv$$

Example 4.7. $u(x) = |x|^{-a}$, $a \in \mathbb{R}$ then $w(x) = \partial_{x_i} u(x) = -a|x|^{-a-2}x_i$ if $a < n-1$. Then $u \in L^1_{loc}(\mathbb{R}^n) \iff a < n$ and $w \in L^1_{loc}(\mathbb{R}^n)$ and $w \in L^1_{loc}(\mathbb{R}^n) \iff a < n-1$. $\forall v \in C_c^\infty(\mathbb{R}^n)$ we have that:

$$\int_{\mathbb{R}^n \setminus B(0,\epsilon)} u\partial_{x_i} v = -\int_{\mathbb{R}^n \setminus B(0,\epsilon)} v\partial_{x_i}(|x|^{-a}) - \int_{\partial B(0,\epsilon)} v|x|^{-a}\nu_i(x)dS(x)$$

Moreover,

$$|\int_{\partial B(0,\epsilon)} v|x|^{-a}\nu_i(x)dS(x)| \leq (\max|v|)(\int_{\partial B(0,1)} 1)\epsilon^{n-1-a} \to 0 \text{ as } \epsilon \to 0$$

since $a < n-1$. We estimate:

$$|\int_{B(0,\epsilon)} u\partial_{x_i} v| \leq \max|\partial_{x_i} v| \int_{B(0,\epsilon)} |u| \implies \int |u| < \infty$$

$$|-\int_{B(0,\epsilon)} v\partial_{x_i}(|x|^{-a})| \leq \max|v| \int_{B(0,\epsilon)} |w| \implies \int |w| < \infty$$

So we apply the dominated convergence theorem, and both tend to 0.

Proposition 4.8. Classical derivatives are weak derivatives.

Proof. Direct integration by parts. □

Proposition 4.9. When it exists, the weak derivatives are unique a.e.

Proof. Assume that $D^\alpha u = w_1$, and $D^\alpha u = w_2$ Then $\forall v \in C_c^\infty(U)$ we have that:

$$\int_U w_1 v = \int_U D^\alpha uv = \int_U w_2 v$$

Then $w_1 - w_2 = 0$ a.e. The result then follows from the next proposition. □

Proposition 4.10. If $u \in L^1_{loc}(U)$ is such that $\forall v \in C_c^\infty(U)$, if $\int_U uv = 0$ then $u = 0$ a.e. in U.

Proof. We use mollifiers to prove this proposition. It follows from (iii) of the next proposition. □

Proposition 4.11. Let $(\eta_\epsilon)_\epsilon$ be a family of non-negative mollifiers in $\mathbb{R}^n$. Then

(i) For $p \in [1,\infty]$ and $u \in L^p_{loc}(U)$, $u \star \eta_\epsilon = C^\infty(U_\epsilon)$ where $U_\epsilon = \{x \in U : d(x,\partial U) > \epsilon\}$.

(ii) For every $p \in [1,\infty]$ and $u \in L^p(U)$, $u \star \eta_\epsilon \in L^p(U_\epsilon)$, we have

$$||u \star \eta_\epsilon||_{L^p(U_\epsilon)} \leq ||u||_{L^p(U)}$$

(iii) For every $p \in [1,\infty)$, $u \in L^p_{loc}(U)$, $u \star \eta_\epsilon \to u$ in $L^p_{loc}(U)$ and a.e. in U.

Proof. (i) $u \star \eta_\epsilon(x) = \int_{B(x,\epsilon)} \eta_\epsilon(x-y)u(y)dy \quad \forall x \in U_\epsilon$. This is the same proof as for when $u \in C^0(U)$.

(ii) We split this into two cases. Case $p = \infty$:

$$|u \star \eta_\epsilon(x) \leq ||u||_{L^\infty(U)} \int_{B(x,\epsilon)} \eta_\epsilon(x-y) = ||u||_{L^\infty(U)}$$

Case $p \in [1, \infty)$:

$$|u \star \eta_\epsilon(x)|^p \leq \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u(y)| \right)^p$$

write $\eta_\epsilon(x-y) = \eta_\epsilon(x-y)^{1/p'}\eta_\epsilon(x-y)^{1/p}$ and apply Holders Inequality

$$\leq \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)^{p'/p'} \right)^{1/p'} \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)^{1/p}|u(y)|^{1/p} \right)^p$$

$$= \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u(y)|^p$$

By integrating, we obtain that:

$$\begin{aligned}
\int_{U_\epsilon} |u \star \eta_\epsilon|^p &\leq \int_{U_\epsilon} \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u(y)|^p dydx \\
&= \int_U \int_{U_\epsilon \cap B(y,\epsilon)} \eta_\epsilon(x-y)|u(y)|^p dydx \quad \text{by Tonelli} \\
&= \int_U |u(y)|^p \int_{U_\epsilon \cap B(y,\epsilon)} \eta_\epsilon(x-y)dxdy \\
&\leq |u(y)|^p dy
\end{aligned}$$

which is the desired result.

(iii) Let V be a compact subset of U. Let $\tilde{V}$ be another compact subset of U such that $V \subset \overset{\circ}{\tilde{V}}$ (the interior). By density of $C^0(\tilde{V})$ in $L^p(\tilde{V})$, there exists $(u_\delta)_\delta$ in $C^0(\tilde{V})$ such that $u_\delta \to u$ in $L^p(V)$. Then:

$$||u \star \eta_\epsilon - u||_{L^p(V)} \leq ||u \star \eta_\epsilon - u_\delta \star \eta_\epsilon||_{L^p(V)} + ||u_\delta \star \eta_\epsilon - u_\delta||_{L^p(V)} + ||u_\delta - u||_{L^p(V)}$$

We have that $||u_\delta \star \eta_\epsilon - u_\delta||_{L^p(V)} \to 0$ as $\epsilon \to 0$. By using (ii) it follows that $||u \star \eta_\epsilon - u_\delta \star \eta_\epsilon||_{L^p(V)} = ||(u - u_\delta) \star \eta_\epsilon||_{L^p(V)} \leq ||u_\delta - u||_{L^p(V)}$:

$$\limsup_{\epsilon \to 0} ||u \star \eta_\epsilon - u||_{L^p(V)} \leq 2||u - u_\delta||_{L^p(V)} \to 0 \text{ as } \delta \to 0$$

which gives that $\lim_{\epsilon \to 0} ||u \star \eta_\epsilon - u||_{L^p(V)} = 0$ i.e. $u \star \eta_\epsilon \to u$ in $L^p(U)$. Note that by Riez-Fischer Theorem, it follows that there exists a subsequence $(u \star \eta_{\epsilon_k})_{k \in \mathbb{N}}$ such that $u \star \eta_{\epsilon_k} \to u$ a.e. in U. This is sufficient in practice, but we can improve it. Write:

$$\begin{aligned}
|u \star \eta_\epsilon(x) - u(x)| &= |\int_{B(x,\epsilon)} \eta_\epsilon(x-y)(u(y) - u(x))dy| \\
&\leq \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u(y) - u(x)|dy \\
&= \epsilon^{-n} \int_{B(x,\epsilon)} \eta(\frac{x-y}{\epsilon})|u(y) - u(x)|dy
\end{aligned}$$

$$= (\int_{B(0,1)} 1) \fint_{B(x,\epsilon)} \eta(\frac{x-y}{\epsilon})|u(y) - u(x)|dy$$

$$\leq (\int_{B(0,1)} 1) \max \eta \fint_{B(x,\epsilon)} |u(y) - u(x)|dy \to 0 \text{ a.e in } U$$

by Lesbesgue's Differentiation Theorem.

□

Corollary 4.1. $C_c^\infty(U)$ is dense in $L^p(U)$ for $p \in [1, \infty)$.

Proof. Combine the density of $C_c^0(U)$ with (iii) of the previous proposition. □

Definition 4.7. (Sobolev Space) For each $k \in \mathbb{N}$ and $p \in [1, \infty]$, the Sobolev space denoted by:

$$W^{k,p}(U) = \{u \in L^p(U) : D^\alpha u \text{ exist in the weak sense and } D^\alpha u \in L^p(U) \text{ for all } |\alpha| \le k\}$$

Moreover, we define: $\forall u \in W^{k,p}(U)$, $||u||_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha| \le k} ||D^\alpha u||_{L^p(U)}^p \right)^{1/p} & \text{if } p \in [1, \infty) \\ \sum_{|\alpha| \le k} ||D^\alpha u||_{L^\infty(U)} & \text{if } p = \infty \end{cases}$

Definition 4.8. (Hilbert Space) When $p = 2$, we denote $H^k(U) = W^{k,2}(U)$ and $|| \cdot ||_{H^k(U)} = || \cdot ||_{W^{k,2}(U)}$, and we define $\forall u \in H^k(U)$:

$$< u, v >_{H^k(U)} = \sum_{|\alpha| \le k} < D^\alpha u, D^\alpha v >_{L^2(U)}$$

so that $||u||_{H^k(U)} = \sqrt{< u, u >_{H^k(U)}}$.

Definition 4.9. (Sobolev Loc Space) We define

$$W^{k,p}_{loc}(U) = \{u \in L^p_{loc}(U) : u \in W^{k,p}(V) \forall \text{ bounded open } V \subset U\}$$

Example 4.8. $u(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } x > 0 \end{cases}$ Then $u \in W^{1,\infty}_{loc}(\mathbb{R})$.

Example 4.9. $u(x) = |x|^{-a}$, $a \in \mathbb{R}$ then $u \in W^{k,p}_{loc}(\mathbb{R}^n)$ if $(a + k)p < n$ and $u \in W^{k,p}(\mathbb{R}^n \setminus B(0, 1))$ if $ap > n$.

Proposition 4.12. For each $k \in \mathbb{N}$ and $p \in [1, \infty]$, $(W^{k,p}(U), || \cdot ||_{W^{k,p}(U)})$ is a Banach space and $(H^k(U), < \cdot, \cdot >_{H^k(U)})$ is a Hilbert space. Moreover, if $u_j \to u$ in $W^{k,p}(U)$ then there exists a subsequence $(u_{\varphi(j)})_j$ such that $u_{\varphi(j)} \to u$ a.e. in $U \quad \forall |\alpha| \le k$.

Proof. We prove the completeness property. The other properties follow from the analog results for L^p spaces. Let $(u_j)_j$ be a cauchy sequence in $W^{k,p}(U)$. Then since $||D^\alpha u_j - D^\alpha u_{j'}||_{L^p(U)} \le ||u_j - u_{j'}||_{W^{k,p}(U)}$, we obtain that $(D^\alpha u_j)_j$ is a cauchy sequence in $L^p(U)$ for each $|\alpha| \le k$. By completeness of $L^p(U)$, it follows that $D^\alpha u_j \to w_\alpha$ in $L^p(U)$. Denote $w_0 = w_(0, \ldots, 0)$. We need to show that $D^\alpha w_0 = w_\alpha$. For each $j \in \mathbb{N}$, we have $\forall v \in C_c^\infty(U)$:

$$\int_U v D^\alpha u_j = (-1)^{|\alpha|} \int_U D^\alpha v u_j \quad (\star)$$

Moreover,

$$\begin{aligned} |\int_U v D^\alpha u_j - \int_U v w_\alpha| &\le |\int_U v(D^\alpha u_j - w_\alpha)| \\ &\le \underbrace{||v||_{L^{p'}(U)}}_{<\infty} \underbrace{||D^\alpha u_j - w_\alpha||_{L^p(U)}}_{\to 0} \end{aligned}$$

$$
\begin{aligned}
|\int_U u_j D^\alpha v - \int_U w_0 D^\alpha u| &\le |\int_U (u_j - w_\alpha) D^\alpha v| \\
&\le \underbrace{||u_j - w_\alpha||_{L^p(U)}}_{\to 0} \underbrace{||D^\alpha v||_{L^{p'}(U)}}_{<\infty}
\end{aligned}
$$

Letting $j \to \infty$ in $(\star)$, we obtain $\forall v \in C_c^\infty(U)$ that:

$$\int_U v w_\alpha = (-1)^{|\alpha|} \int_U D^\alpha v w_0$$

Thus $w_\alpha = D^\alpha w_0$ as desired. □

Proposition 4.13. Assume that U is bounded, $1 \le k_1 \le k_2$ and $1 \le p_1 \le p_2 \le \infty$. Then $\exists c > 0$ such that $\forall u \in W^{k_2,p_2}(U)$,

$$||u||_{W^{k_1,p_1}(U)} \le c||u||_{W^{k_2,p_2}(U)}$$

In particular, $W^{k_2,p_2}(U) \subset W^{k_1,p_1}(U)$.

Proof. Follows directly from the analog result for L^p spaces. □

Proposition 4.14. Let $k \in \mathbb{N}$, $p \in [1,\infty]$, $\eta \in C^\infty(U) \cap W^{k,\infty}(U)$ and $u \in W^{k,p}(U)$. Then $\eta u \in W^{k,p}(U)$ and the product rule applies. In particular, $D(\eta u) = \eta Du + D\eta u$.

Proof. We prove the case $k = 1$. For $k \ge 2$, follows by induction. $\forall v \in C_c^\infty(U)$, we have that:

$$\int_U v(\eta Du + uD\eta) + \int_U \eta u Dv = \int_U u(vD\eta + \eta Dv) + \int_U \eta v Du = 0 \quad \text{since } v\eta \in C_c^\infty(U)$$

Hence $\int_U v(\eta Du + uD\eta) = -\int_U \eta u Dv$ so $D(\eta u) = D\eta u + Du\eta$.
Moreover,

$$||\eta u||_{L^p(U)} \le ||\eta||_{L^\infty(U)} ||u||_{L^p(U)}$$

and

$$||\eta Du + D\eta u||_{L^p(U)} \le ||\eta||_{L^\infty(U)} ||Du||_{L^p(U)} + ||D\eta_{L^\infty(U)}||u||_{L^p(U)}$$

hence ηu and $\eta Du + D\eta u \in L^p(U)$ as desired. □

Proposition 4.15. Let $(\eta_\epsilon)_\epsilon$ be a family of mollifiers in $\mathbb{R}^n$.

(i) If $p \in [1,\infty]$ and $u \in W^{k,p}(U)$, $D^\alpha(\eta_\epsilon \star u) = \eta_\epsilon \star D^\alpha u$

(ii) If $p \in [1,\infty]$ and $u \in W^{k,p}(U)$, $||\eta_\epsilon \star u||_{W^{k,p}(U)} \le ||u||_{W^{k,p}(U)}$

(iii) If $p \in [1,\infty)$ and $u \in W^{k,p}_{loc}(U)$, $\eta_\epsilon \star u \to u$ in $W^{k,p}_{loc}(U)$ and a.e. in U

Proof. We only need to prove (i) since (ii) and (iii) follow directly from our analog in L^p spaces.

$$
\begin{aligned}
D^\alpha(\eta_\epsilon \star u)(x) &= \int_{B(x,\epsilon)} D_x^\alpha(\eta_\epsilon(x-y)u(y)dy) \\
&= (-1)^{|\alpha|} \int_{B(x,\epsilon)} D_y^\alpha \eta_\epsilon(x-y)u(y)dy \\
&= \int_{B(x,\epsilon)} \eta_\epsilon(x-y) D^\alpha u(y)dy \\
&= \eta_\epsilon \star D^\alpha u(x)
\end{aligned}
$$

□

Remark 4.2. The approximation by functions in $C_c^\infty(U)$ of functions in $W^{k,p}_{loc}(U)$ can be improved: It can be shown that if U is bounded then $C^\infty(U) \cap W^{k,p}(U)$ is dense in $W^{k,p}(U)$, and if moreover ∂U is C^1 then $C^\infty(\overline{U})$ is dense in $W^{k,p}(u)$ for $p \in [1, \infty)$. Evan's book elaborates more on this in sections 5.3.2 and 5.3.3.

Proposition 4.16. Assume that U is open and connected and $u \in W^{k,p}(U)$ such that $Du \equiv 0$ a.e. in U. Then u is constant a.e. in U.

Proof. This is Q4 of A4. ☐

Definition 4.10. Let $k \in \mathbb{N}$ and $p \in [1, \infty]$. We define the Sobolev space $W_0^{k,p}(U)$ as the closure of $C_c^\infty(U)$ in $(W^{k,p}(U), ||\cdot||_{W^{k,p}(U)})$, i.e. the subspace of all $u \in W^{k,p}(U)$ such that there exists $(u_j)_j$ in $C_c^\infty(U)$ such that $u_j \to u$ in $W^{k,p}(U)$.

Proposition 4.17. $W_0 k, p(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$.

Proof. Let $u \in W^{k,p}(\mathbb{R}^n)$. Define for each $\delta > 0$, $u_\delta(x) = \eta_\delta(x) u(x)$ such that $\eta_\delta(x) = \eta(\delta x)$ for some cutoff function $\eta \in C^\infty(\mathbb{R})$ such that

$$\eta = \begin{cases} 1, & \text{in } B(0,1) \\ 0, & \text{in } \mathbb{R}^n \setminus B(0,2) \end{cases}$$

And $0 \leq \zeta \leq 1 \in B(0,2) \setminus B(0,1)$. Then $u_\delta \in W^{k,p}(\mathbb{R}^n)$ and the product rule gives for some $c > 0$ independent of δ:

$$\begin{aligned} ||u_\delta - u||_{W^{k,p}(\mathbb{R}^n)} &\leq c \sum_{|\alpha| \leq k} ||D^\alpha(\zeta_\delta - 1) D^\beta u||_{L^p(\mathbb{R}^n)} \\ &\leq c \sum_{|\alpha| \leq k} ||D^\alpha(\zeta_\delta - 1)||_{L^\infty(\mathbb{R}^n)} ||D^\beta u||_{L^p(\mathbb{R}^n \setminus B(0, \frac{1}{\delta}))} \end{aligned}$$

Note that

$$||D^\alpha(\zeta_\delta - 1)||_{L^\infty(\mathbb{R}^n)} = \begin{cases} 1, & \text{if } |\alpha| = 0 \\ \delta^{|\alpha|} ||D^\alpha \zeta||_{L^\infty(\mathbb{R}^n)} & \text{if } |\alpha| \geq 1 \end{cases}$$

We also have that $||D^\beta u||_{L^p(\mathbb{R}^n \setminus B(0, \frac{1}{\delta}))} \to 0$ as $\delta \to 0$ by the Dominated Convergence Theorem since $D^\beta u \in L^p(\mathbb{R}^n)$. Hence $u_\delta \to u$ in $W^{k,p}(\mathbb{R}^n)$. Moreover, for each $\delta > 0$ since $u_\delta = 0$ in $\mathbb{R}^n \setminus B(0,2)$, letting $(\eta_\epsilon)_\epsilon$ be a family of mollifiers in $\mathbb{R}^n$, we have that $\eta_\epsilon \star u_\delta \to u_\delta$ in $W^{k,p}(\mathbb{R}^n)$ hence there exists a subsequence $(\epsilon_\delta)_\delta$ such that $\eta_{\epsilon_\delta} \star u_\delta \to u$ in $W^{k,p}(\mathbb{R}^n)$ which gives that $u \in W_0^{k,p}(\mathbb{R}^n)$ as $u_\delta \in C_c^\infty(\mathbb{R}^n)$. ☐

Theorem 4.7. (Poincare's Inequality) Assume that U is bounded, $k \in \mathbb{N}$, and $p \in [1, \infty)$. Then there exists $c > 0$ such that $\forall u \in W_0^{k,p}(U)$,

$$||u||_{W^{k,p}(U)} \leq c ||u||_{W_0^{k,p}(U)}$$

where $||u||_{W_0^{k,p}(U)} = \left(\sum_{|\alpha| \leq k} ||D^\alpha u||^p_{L^p(U)} \right)^{1/p}$. In particular, $||\cdot||_{W_0^{k,p}(U)}$ is an equivalent norm to $||\cdot||_{W^{k,p}(U)}$.

Proof. By density of $C_c^\infty(U)$ in $W_0^{k,p}(U)$, it suffices to establish the inequality with $u \in C_c^\infty(U)$.

Case $k = 1$: $\forall x \in U$,

$$|u(x_1, \dots, x_n)|^p = |\int_{\underline{x_1}}^{x_1} \partial_{x_1} u(s, x_2, \dots, x_n) ds|^p$$
$$\leq \left[(\int_{\underline{x_1}}^{x_1} 1 ds)^{1/p'} (\int_{\underline{x_1}}^{x_1} |\partial_{x_1} u(s, x_2, \dots, x_n)|^p ds)^{1/p} \right]^p$$
$$\leq \left[(\overline{x_1} - \underline{x_1})^{1/p'} (\int_{\underline{x_1}}^{x_1} |\partial_{x_1} u(s, x_2, \dots, x_n)|^p ds)^{1/p} \right]^p$$

By integrating,

$$\int_{\mathbb{R}^n} |u|^p \leq (\overline{x_1} - \underline{x_1})^{p/p'} \int_{\mathbb{R}^n} \int_{\underline{x_1}}^{x_1} |\partial_{x_1} u(s, x_2, \dots, x_n)|^p (ds) d(x_1, \dots, x_n)$$
$$= (\overline{x_1} - \underline{x_1})^{p/p'} \int_{\mathbb{R}^{n-1}} \int_{\underline{x_1}}^{\overline{x_1}} \int_{s}^{\overline{x_1}} |\partial_{x_1} u(s, x_2, \dots, x_n)|^p (dx_1) d(s, x_2, \dots, x_n)$$
$$\leq (\overline{x_1} - \underline{x_1})^{p/p'+1=p} \int_{\mathbb{R}^n} |\partial_{x_1} u|^p$$

Case $k \geq 2$: By induction, $\forall u$ we have:

$$||D^\alpha u||_{L^p(U)} \leq (\overline{x_1} - \underline{x_1})^{k-|\alpha|} ||\partial_{x_1}^{k-|\alpha|} D^\alpha u||_{L^p(U)}$$

gives $||u||_{W^{k,p}(U)} \leq c||u||_{W_0^{k,p}(U)}$. □

Example 4.10. (A Counter Example to Poincare's Inequality) Let $U = \mathbb{R}^n$ and $u \in C_c^\infty(\mathbb{R})$. Set $u_\delta(x) = u(\delta x) \quad \forall x \in \mathbb{R}^n, \delta > 0$. Then we have that:

$$\frac{\int_{\mathbb{R}^n} |Du_\delta|^p}{\int_{\mathbb{R}^n} |u_\delta|^p} = \delta^p \frac{\int_{\mathbb{R}^n} |Du|^p}{\int_{\mathbb{R}^n} |u|^p} \to 0 \text{ as } \delta \to 0$$

Now consider the problem $\begin{cases} Lu = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$ where U is an open, bounded subset of $\mathbb{R}^n$ and

$$L = -\sum_{i,j=1}^n \partial_{x_1}(a_{ij}(x)\partial_{x_i} u) + \sum_{i=1}^n b_i(x)\partial_{x_i} u + c(x)u \quad (\star)$$

for $a_{i,j}, b_i \in L^\infty(U)$, $a_{ij} = a_{ji}$, $f \in L^2(U)$, and $g \in H^1(U)$.

Definition 4.11. (Weak Solution) We define $H_g^1(U) = \{u + g : u \in H_0^1(U)\}$. We say that $u : U \to \mathbb{R}$ is a weak solution of $(\star)$ if $u \in H_g^1(U)$ and $\forall v \in C_c^\infty(U)$:

$$\int_U \sum_{i,j=1}^n a_{ij}(\partial_{x_i} u)(\partial_{x_j} v) + \sum_{i=1}^n b_i(\partial_{x_i} u)v + cuv = \int_U fv$$

Remark 4.3. (i) Holders Inequality guarantees that the integrals are finite

(ii) Assuming that $g \in H^1(U)$ is not a restriction, as we will see by the next proposition

Proposition 4.18. Let U be an open bounded subset of $\mathbb{R}^n$ such that ∂U is C^k. Then for every $g \in C^k(\partial U)$ there exists $\tilde{g} \in C^k(\overline{U})$ such that $\tilde{g} = g$ on ∂U.

Proof. Since ∂U is compact, $\forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N}$, $x_1, \dots, x_{N_\epsilon} \in \partial U$, $B(x_1, \epsilon), \dots, B(x_{N_\epsilon}, \epsilon)$ such that $\partial U \subset \bigcup_{i=1}^{N_\epsilon} B(x_i, \epsilon)$. Moreover, by using strengthening coordinates, $\exists \phi_i : U_i \to \phi_i(U)$ is C^k diffeomorphism such that

$$\phi_i(U_i \cap U) = \phi_i(U_i) \cap \mathbb{R}^{n-1} \times (0, \infty)$$

$$\phi_i(U_i \cap \partial U) = \phi_i(U_i) \cap \mathbb{R}^{n-1} \times \{0\}$$

and $B(x_i, \epsilon) \subset U_i$ (possible $\phi_i = \phi_j$ so that U_i does not depend on ϵ). By letting $\epsilon > 0$ be small enough, we obtain that the projection

$$\pi_n(\phi_i(B(x_i, 2\epsilon))) \subset \phi_i(U_i)$$

where $\pi_n(x_1, \dots, x_n) = \pi_n(x_1, \dots, x_{n-1}, 0)$. Now define:

$$\tilde{g}(x) = \zeta(x) \sum_{i=1}^{N_\epsilon} \frac{\rho(\frac{|x-x_i|}{\epsilon})}{\sum_{j=1}^{N_\epsilon} \rho(\frac{|x-x_j|}{\epsilon})} g(\phi_i^{-1}(\pi_n(\phi_i(x)))) \quad x \in \overline{U}$$

for some cutoff functions $\rho \in C^\infty([0, \infty))$ such that $\rho = 1$ on $[0, 1]$, $\rho = 0$ on $[2, \infty]$, and $0 \leq \rho \leq 1$ on $[1, 2]$, and $\zeta \in C^\infty(\mathbb{R}^n)$ such that $\zeta = 1$ on $\{x \in \mathbb{R}^n : d(x, \partial U) \leq \delta\}$ and $\zeta = 0$ on $\{x \in \mathbb{R}^n : d(x, 0) \geq 2\delta\}$. δ is chosen so that $\{x \in \mathbb{R}^n : d(x, \partial U) \leq 2\delta\} \subset \bigcup_{i=1}^{N_\epsilon} B(x_i, \epsilon)$. Then we have that $\sum_{j=1}^{N_\epsilon} \rho(\frac{|x-x_j|}{\epsilon}) > 0$ when $\zeta(x) > 0$. This implies that $\tilde{g} \in C^k(\overline{U})$. Moreover, $g = \tilde{g}$ on ∂U. ☐

Remark 4.4. It can be extended in more generality with the Whitney Extension Theorem.

We used the following for our previous proof:

Lemma 4.1. Let K be compact and $U \subset \mathbb{R}^n$ be open such that $K \subset U$. Then $\exists \zeta \in C_c^\infty(\mathbb{R}^n)$ such that $\zeta = 1$ on K and $\zeta = 0$ on $\mathbb{R}^n \setminus U$.

Proof. We will use a mollifiers argument. Let (η_ϵ) be a family of mollifiers in $\mathbb{R}^n$. Then define $\zeta(x) = \eta_\epsilon \star \chi_{K_\epsilon}$ where $K_\epsilon = \{x \in \mathbb{R}^n : d(x, K) < \epsilon\}$ and $\chi_{K_\epsilon} = \begin{cases} 1, & \text{if } x \in K_\epsilon \\ 0, & \text{if } x \notin K_\epsilon \end{cases}$ If ϵ is chosen small enough, then $\zeta = 1$ on K and $\zeta \in C_c^\infty(U)$. ☐

Proposition 4.19. Assume that U is bounded, ∂U is C^1, $a_{ij} \in C^1(U)$, $b_i \in C^0(U)$, $g \in H^1(U) \cap C^0(\overline{U})$ and $u \in C^2(U) \cap H^1(U) \cap C^0(\overline{U})$. Then u is a weak solution of:

$$\begin{cases} Lu = -\sum\limits_{i,j=1}^{n} \partial_{x_1}(a_{ij}(x)\partial_{x_i}u) + \sum\limits_{i=1}^{n} b_i(x)\partial_{x_i}u + c(x)u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

$\iff$ it is a classical solution.

Proof. By integrating by parts, since $u \in C^2(U)$, $a_{ij} \in C^1(U)$, $b_i \in C^\infty(U)$, and $f \in C^0(U)$ we obtain that $\forall v \in C_c^\infty(U)$:

$$\int v(Lu - f) = \int v(\sum_{i,j=1}^{n} a_{ij}(\partial_{x_i}u)(\partial_{x_i}v) + \sum_{i=1}^{n} b_i(\partial_{x_i}u)v + cuv - fv) \quad (\star)$$

Hence $\forall v \in C_c^\infty(U)$ $(\star) = 0 \iff \int_U v(Lu - f) = 0 \forall v \in C_c^\infty(U) \iff Lu = f$ in U. It remains to show that $u \in H_g^1(U) \iff u = g$ on ∂U, i.e. $u - g \in H_0^1(U)$, i.e. $u - g = 0$ on ∂U. ☐

More generally, we prove:

Proposition 4.20. Assume that $p \in [1,\infty)$, ∂U is C^1 and $u \in W^{1,p}(U) \cap C^0(\overline{U})$. Then $u \in W_0^{1,p}(U) \iff u = 0$ on ∂U.

Proof. First, suppose that $u = 0$ on ∂U. If U is unbounded, we can approximate u by $u_\delta = \eta_\delta u$ where η_δ is as in the proof of $W^{1,p}(\mathbb{R}^n) = W_0^{1,p}(\mathbb{R}^n)$. This reduces the problem to the case of compactly supported u.

Let $\rho \in C_c^\infty(\mathbb{R})$ be a cutof function such that $\rho = 1$ on $[-1, 1]$, $\rho = 0$ on $\mathbb{R} \setminus [-2, 2]$, and $0 \le \rho \le 1$. Define $F(t) = (1 - \rho(t))t$ for all $t \in \mathbb{R}$. Finally, define $u_\epsilon(x) = \epsilon F(\epsilon^{-1}u(x))$ for all $\epsilon > 0$ and $x \in \overline{U}$. By Q5 of A4, we have that $u_\epsilon \in W^{1,p}(U)$ and $Du_\epsilon = F'(\epsilon^{-1}u)Du$.

If $|u(x)| \ge 2\epsilon$, then $u_\epsilon(x) = u(x)$. Moreover, $|u_\epsilon| \le \max|1 - \rho| \cdot |u| \in L^p(U)$ and $|Du_\epsilon| \le \max|F'| \cdot |Du| \in L^p(U)$. By the dominated convergence theorem, $u_\epsilon \to u$ in $W^{1,p}(U)$.

The closure of $\{x \in U : u_\epsilon(x) \neq 0\}$ is contained in $\{x \in K : |u(x)| \ge \epsilon\}$, where K contains the support of U. This set is compact and disjoint from ∂U by continuity. We can thuse approximate u_ϵ by functions in $C_c^\infty(U)$ using mollifiers. This proves that $u \in W_0^{1,p}(U)$.

Now, assume that $u \in W_0^{1,p}(U)$. Fix $x \in \partial U$. We multiply u by a sufficiently small cutoff function to use straightening coordinates. By the change of variables formula for integration, we can transfer the problem to a flat boundary. Then u is zero outside of a compact set K such that $U \cap K \subset \mathbb{R}^n \times (0,\infty)$ and $\partial U \cap K \subset \mathbb{R}^{n-1} \times \{0\}$, where the point x is now the origin.

By taking small enough $\epsilon, \delta > 0$, we can work in $[-\epsilon,\epsilon]^{n-1} \times [-\delta,\delta] \subset K$. Since $u \in W_0^{1,p}(U)$, there exists $(u_k)_{k\in\mathbb{N}}$ in $C_c^\infty(U)$ such that $u_k \to u$ in $W^{1,p}(U)$. By observing that $||\cdot||_{W^{1,p}((-\epsilon,\epsilon)^{n-1}\times(0,\delta))} \le c||\cdot||_{W^{1,p}(U)}$ for some constant c, we have $u_k \to u$ in $W^{1,1}((-\epsilon,\epsilon)^{n-1} \times (0,\delta))$. As in the proof of Poincare's Inequality, we have by writing $u_k(x) = u_k(x) - u_k(x_1,\dots,x_{n-1},0)$ and using the fundamental theorem of calculus that

$$\int_{(-\epsilon,\epsilon)^{n-1}\times(0,\delta)} |u_k| \le \delta \int_{(-\epsilon,\epsilon)^{n-1}\times(0,\delta)} |Du_k|$$

Letting k go to infinity yields the same inequality but for u. Rewriting, we have

$$\frac{1}{\delta}\int_0^\delta \int_{(-\epsilon,\epsilon)^{n-1}} |u|(dx_1\dots dx_{n-1})dx_n \le \int_{(-\epsilon,\epsilon)^{n-1}\times(0,\delta)} |Du|$$

As $\delta \to 0$, the right side goes to zero by the dominated convergence theorem, as the domain of integration tends to zero. The left side, however, is

$$⨍_{(0,\delta)} \int_{(-\epsilon,\epsilon)^{n-1}} |u|(dx_1\dots dx_{n-1})dx_n \to \int_{(-\epsilon,\epsilon)^{n-1}} |u|$$

by continuity of u So the integral of $|u|$ on this portion of the boundary is zero and thus u is zero almost everywhere on $(-\epsilon,\epsilon) \times \{0\}$. By continuity of u it is genuinely zero. So $u(x) = 0$. Thus $u = 0$ on ∂U. ☐

4.4 Existence and Regularity of Weak Solutions

Theorem 4.8. (Riesz Representation Theorem (RRT))
Let $< H, < \cdot,\cdot >_H >$ be a Hilbert space. Let $F : H \to \mathbb{R}$ be linear and bounded. Then there exists a unique $u \in H$ such that for all $v \in H$, we have $< u, v >= F(v)$. Moreover,

$$\frac{1}{2}||u||_H^2 - F(u) = \min_{v\in H}(\frac{1}{2}||v||_H^2 - F(v))$$

Proof. Since F is bounded by some constant c, we have that for all $v \in H$ that:

$$\frac{1}{2}||u||_H^2 - F(u) \ge \frac{1}{2}||v||_H^2 - c||v||_H \ge -\frac{c^2}{2} > -\infty$$

so there exists $(u_k)_{k\in\mathbb{N}}$ such that

$$\frac{1}{2}||u_k||_H^2 - F(u_k) \to \lambda := \inf_{v\in H}\left(\frac{1}{2}||v||_H^2 - F(v)\right)$$

By the parallelogram law,

$$||\frac{u_k - u_{k'}}{2}||^2 + ||\frac{u_k + u_{k'}}{2}||^2 = \frac{1}{2}||u_k||^2 + \frac{1}{2}||u_{k'}||^2$$

Then

$$\begin{aligned}||u_k - u_{k'}||^2 &= 2(||u_k|| + ||u_{k'}||)^2 - 4||\frac{u_k + u_{k'}}{2}||^2 \\ &= 4(\underbrace{\frac{1}{2}||u_k||^2 - F(u_k)}_{\to\lambda} + \underbrace{\frac{1}{2}||u_{k'}||^2 - F(u_{k'})}_{\to\lambda}) - \underbrace{8(\frac{1}{2}||\frac{u_k + u_{k'}}{2}||^2 - F(\frac{u_k + u_{k'}}{2}))}_{\geq\lambda}\end{aligned}$$

where $\lambda = \min_{v\in H}(\frac{1}{2}||v||_H^2 - F(v))$. This implies that $\forall\epsilon > 0$, $\exists k_\epsilon \in \mathbb{N}$ such that $\forall l, k' \geq k_\epsilon$,

$$||u_k - u_{k'}||^2 \leq 4(\lambda + \frac{\epsilon}{8} + \lambda + \frac{\epsilon}{8}) - 8\lambda = \epsilon$$

Thus $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in H. By completeness, it follows that $u_k \to u$ in H. By continuity of $||\cdot||_H$ and F we obtain $\frac{1}{2}||u||_H^2 - F(u) = \lambda$.

Now, $\forall v \in H$, $t > 0$ we have that

$$\begin{aligned}\frac{1}{2}||u||_H^2 - F(u) &\leq \frac{1}{2}||u + tv||_H^2 - F(u + tv) \\ &= \frac{1}{2}||u||_H^2 - F(u) + t(< u, v >_H - F(v)) + \frac{t^2}{2}||v||_H^2\end{aligned}$$

So we have that $0 \leq < u, v >_H - F(v) + \frac{t}{2}||v||_H^2$. Now, as $t \to 0$, we have that $F(v) \leq < u, v >_H$.

Considering $-v$ instead of v, we obtain $F(v) = -F(-v) \geq - < u, -v >_H = < u, v >_H$ so $F(v) = < u, v >_H \quad \forall v \in H$. □

Theorem 4.9. (Lax-Milgram Theorem (LMT))
Let $< H, < \cdot, \cdot >_H >$ be a Hilbert space. Let $B : H \times H \to \mathbb{R}$ and $F : H \to \mathbb{R}$ be such that:

(i) B is bilinear

(ii) B is continuous (equivalently, bounded): there exists $c_1 > 0$ such that $|B(u, v)| \leq c_1||u||_H \cdot ||v||_H$ for all $u, v \in H$.

(iii) B is coercive: there exists $c_2 > 0$ such that $B(u, u) \geq c_2||u||_H^2$ for all $u \in H$.

(iv) F is linear

(v) F is continuous (equivalently, bounded): there exists $c_3 > 0$ such that $|F(u)| \leq c_3||u||_H$ for all $u \in H$.

Then there exists a unique solution to $u \in H$ of the equation $B(u, \cdot) = F$ in H. Moreover, if B is symmetric, then:

$$\frac{1}{2}B(u, u) - F(u) = \min_{v\in H}(\frac{1}{2}B(v, v) - F(v))$$

Proof. We will first begin with the case where B is symmetric: In this case, B is an inner product on H and its induced norm is equivalent to $||\cdot||_H$. Indeed, $\forall u \in H$,

$$c_1||u||_H^2 \geq B(u,u) \geq c_2||u||_H^2 \geq 0 \text{ s.t. } = 0 \iff u = 0$$

Then the LMT follows from the RRT applied to (H, B).

Now we consider the general case: By using RRT, since F and $B(u,\cdot)$ are linear and bounded, $\forall u$ we obtain that $\exists! f, A(u) \in H$ such that $F =< f, \cdot >_H$ and $B(u,\cdot) =< A(u), \cdot >_H$. We want to show that $\exists! u \in H$ such that $A(u) = f$. Since $B(\cdot, v)$ is linear $\forall v$ we obtain that A is linear. Moreover, A is bounded $\forall u \in H$:

$$||A(u)||^2 = B(u, A(u)) \leq c_1||u||_H||A(u)||_H \implies ||A(u)||_H \leq c_1||u||_H$$

Now, we will show that A is bijective. First injectivity: $\forall u \in \ker(A)$,

$$0 =< A(u), u >= B(u,u) \geq c_2||u||_H^2 \implies u = 0$$

Now surjectivity: $\forall u \in A(H)^\perp$,

$$0 =< A(u), u >= B(u,u) \geq c_2||u||_H^2 \implies u = 0$$

Thus A is bijective, so $\forall f \in H \exists! u \in H$ such that $A(u) = f$. ☐

Definition 4.12. (Uniformly Elliptic)
Let L be such that $u \mapsto Lu = \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x)\partial_{x_i}u) + \sum_{i=1}^n b_i(x)\partial_{x_i}u + c(x)u$, where $a_{ij}, b_i, c \in L^\infty(U)$, $a_{ij} = a_{ji}$. We say that L is uniformly elliptic in U if there exists a constant $C > 0$ such that $\forall x \in U$, $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n$,

$$\sum_{i,j=1}^n a_{ij}(x)\zeta_i\zeta_j \geq C|\zeta|^2$$

In particular, $A(x) = (a_{ij}(x))_{ij}$ is positive-definite (it has positive eigenvalues).

Remark 4.5. The wave equation is hypebolic, the heat equation is parabolic, and the Laplace equation is elliptic.

Proposition 4.21. Assume that U is bounded, $a_{ij}, b_i, c \in L^\infty(U)$, $a_{ij} = a_{ji}$, L is uniformly elliptic in U, and $c \geq 0$, $f \in L^2(U)$, and $g \in H^1(U)$. Then there exists $\epsilon > 0$ depending only on U and the constant of ellipticity such that if $\sum_{i=1}^n ||b_i||_{L^\infty(U)} \leq \epsilon$ then $\exists!$ a weak solution u of

$$\begin{cases} Lu = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases} \quad (\star)$$

Proof. Observe that $u \in H^1(U)$ is a weak solution of $(\star) \iff \tilde{u} = u - g \in H_0^1(U)$ and $B(\tilde{u}, v) = F(v) \quad \forall v \in C_c^\infty(U)$ where $\forall u, v$,

$$B(u,v) = \int_U \sum_{i,j=1}^n a_{ij}(\partial_{x_i}u)(\partial_{x_i}v) + \sum_{i=1}^n b_i(\partial_{x_i}u)v + cuv$$

and

$$F(v) = \int_U fv - \sum_{i,j=1}^n a_{ij}(\partial x_i g)(\partial x_j v) - \sum_{i=1}^n b_i(\partial_{x_i}g)v - cgv$$

By Holders Inequality, clearly F is linear and B is linear, and we have that:

$$\begin{aligned}|B(u,v)| &\le \sum_{i,j=1}^{n} ||a_{ij}||_{L^\infty(U)}||\partial x_i u||_{L^2(U)}||\partial x_j v||_{L^2(U)} + \sum_{i=1}^{n} ||b_i||_{L^\infty(U)}||\partial x_i u||_{L^2(U)}||v||_{L^2(U)} \\ &\quad + ||c||_{L^\infty(U)}||u||_{L^2(U)}||v||_{L^2(U)} \\ &\le (\sum_{i,j=1}^{n} ||a_{ij}||_{L^\infty(U)} + \sum_{i=1}^{n} ||b_i||_{L^\infty(U)}) + ||c||_{L^\infty(U)})||u||_{H^1(U)}||v||_{H^1(U)}\end{aligned}$$

Similarily,

$$\begin{aligned}|F(v)| &\le \left(||f||_{L^2(U)} + \sum_{i,j=1}^{n} ||a_{ij}||_{L^\infty(U)}||\partial x_i g||_{L^2(U)}||\partial x_j v||_{L^2(U)} \right. \\ &\quad \left. - \sum_{i=1}^{n} ||b_i||_{L^\infty(U)}||\partial x_i g||_{L^2(U)} + ||c||_{L^\infty(U)}||g||_{L^2(U)} \right) ||v||_{L^2(U)} \\ &\le c||u||_{H_0^1(U)}||v||_{H_0^1(U)}\end{aligned}$$

Now we have that:

$$\begin{aligned}B(u,u) &= \int_U \underbrace{\sum_{i,j=1}^{n} a_{ij}(\partial x_i u)(\partial x_j u)}_{\ge c|Du|^2} + \underbrace{\sum_{i=1}^{n} b_i(\partial x_i u)\, u}_{-\epsilon|Du||u|} + \underbrace{cu^2}_{\ge 0} \\ &\ge ||Du||_{L^2(U)} - \epsilon \int_U |Du||u| \\ &\ge c||u||_{H_0^1(U)} - \epsilon||u||_{H_0^1(U)} \underbrace{||u||_{L^2(U)}}_{\le c'||u||_{H_0^1(U)}}\end{aligned}$$

for some constants c, c' depending only on the ellipticity of L and on U. In particular, if $\epsilon < \frac{c}{c'}$ we obtain that B is coercive. Then the LMT applies and we are done. $\square$

Theorem 4.10. Assume that $a_{ij} \in C^{k+1}(U)$, $a_{ij} = a_{ji}$, $b_i, c \in C^k(U)$, $f \in H^k(U)$ for some $k \in \mathbb{N} \cup \{0\}$, L is uniformly elliptic in U, and u is a weak solution of $Lu = f$ in U. Then $u \in H_{loc}^{k+2}(U)$ and for every open subset V of U such that $\overline{V} \subset U$, there exists a constant $C > 0$ independent of u and f such that

$$||u||_{H^{k+2}(V)} \le C(||u||_{L^2(U)}) + ||f||_{H^k(U)}$$

Proof. We start with the case $k = 0$. Consider $v_{l,\epsilon} = -D_l^{-\epsilon}(\rho^2 D_l^\epsilon u)$ where $\epsilon > 0$ is small, $l \in [n]$, $\rho \in C_c^\infty(U)$ is a cutoff function such that $\rho = 1$ on V and $\rho \in [0,1]$ in U, and

$$D_l^{\pm\epsilon} u = \frac{u(x \pm \epsilon e_l) - u(x)}{\pm\epsilon}, \quad e_l = (\underbrace{0, \ldots, 0}_{l-1}, 1, \ldots, 0)$$

so that $v_{l,\epsilon}$ has compact support in U for ϵ small enough. Since $u \in H^1(U)$, it follows that $v_{l,\epsilon} \in H_0^1(U)$. Since $Lu = f$ in U, we obtain:

$$\int_U \sum_{i,j}^{n} a_{ij}(\partial_{x_i} u)(\partial_{x_j} v_{l,\epsilon}) + (\sum_{i}^{n} b_i(\partial_{x_i} u) + cu - f)v_{l,\epsilon}) = 0$$

which we obtained by writing $v_{l,\epsilon}$ as the limit of $(v_k)_k \subset C_c^\infty(U)$. Then we have that:

$$\sum_{i,j}^n \int_U a_{ij}(\partial_{x_i} u)(\partial_{x_j} v_{l,\epsilon}) = \sum_{i,j}^n \int_U D_l^\epsilon(a_{ij}(\partial_{x_i} u)) D_l^\epsilon(\rho^2 \partial_{x_j} v_{l,\epsilon})$$

This implies that

$$\sum_{i,j}^n \int_U a_{ij}(\partial_{x_i} u)(\partial_{x_j} v_{l,\epsilon}) = \sum_{i,j}^n \int_U [a_{ij}(x+\epsilon e_l) D_l^\epsilon(\partial_{x_i} u) + (D_l^\epsilon a_{ij})\partial_{x_i} u](\rho^2 D_l^\epsilon \partial_{x_j} v_{l,\epsilon} + 2\rho(\partial_{x_j}\rho) D_l^\epsilon u)$$
$$\geq \int_U c\rho^2 |D_l^\epsilon Du|^2 - c'\rho(|D_l^\epsilon Du|)(|D_l^\epsilon u| + |Du|) + (|Du||D_\epsilon^\epsilon u|)$$

where c' depends on $\max_{supp(\rho)}(|a_{ij}| + |Da_{ij}| + |\rho| + |D\rho|)$ but not on u.

By Young's Inequality, $ab \leq \delta a^2 + \frac{b^2}{4\delta} = (\sqrt{\delta} a - \frac{b}{2\sqrt{\delta}})^2 + ab)$. We obtain:

(i) $\rho |D_l^\epsilon Du||D_l^\epsilon u| \leq \delta\rho^2 |D_l^\epsilon Du|^2 + \frac{1}{4\delta}|D_l^\epsilon u|^2$

(ii) $\rho |D_l^\epsilon Du||Du| \leq \delta\rho^2 |D_l^\epsilon Du|^2 + \frac{1}{4\delta}|Du|^2$

(iii) $|Du||D_l^\epsilon u| \leq \frac{1}{2}|Du|^2 + \frac{1}{2}|D_l^\epsilon u|^2$

By choosing δ small enough, we obtain that

$$\sum_{i,j}^n \int_U a_{ij}(\partial_{x_i} u)(\partial_{x_j} v_{l,\epsilon}) \geq c\int_U \rho^2 |D_l^\epsilon Du|^2 - c' \int_{supp(\rho)} |D_l^\epsilon u|^2 + |Du|^2$$

for some constants $c, c' > 0$, (different from the previous estimate) independent of u. Similarly,

$$|\int_U (\sum_i b_i(\partial_{x_i} u) + cu - f) v_{l,\epsilon}| \leq c\int_U |D_l^{-\epsilon}(\rho^2 D_l^\epsilon u)|(|Du| + |u| + |f|)$$
$$\leq \delta \int_U |D_l^{-\epsilon}(\rho^2 D_l^\epsilon u)|^2 + \frac{c'}{\delta}$$
$$\leq \delta \int_U |D_l^{-\epsilon}(\rho^2 D_l^\epsilon u)|^2 + \frac{c'}{\delta}\int_{supp(\rho)} |Du|^2 + |u|^2 + |f|^2$$

for some constant c, c' depending on $\max_{supp(\rho)}(|c| + |b_i|)$ not on u and f. Finally we obtain,

$$\int_U \rho^2 |D_l^\epsilon Du|^2 \leq \delta \int_U |D_l^{-\epsilon}(\rho^2 D_l^\epsilon)|^2 + c_\delta \int_{supp(\rho)} |D_l^\epsilon u|^2 + |Du|^2 + u^2 + f^2$$

for some c_δ independent of u and f. Observe that

$$\int_{supp(\varphi)} |D_l^\epsilon u(x)|^2 = \int_{supp(\varphi)} \left| \int_0^1 \partial x_l u(x + \epsilon s e_l) ds \right|^2 dx$$
$$\leq \int_{supp(\varphi)} \int_0^1 |\partial x_l u(x + \epsilon s e_l)|^2 ds dx$$
$$= \int_0^1 \int_{supp(\varphi)} |\partial x_l u(x + \epsilon s e_l)|^2 dx ds$$
$$\leq \int_0^1 \int_{supp(\varphi)+\epsilon s e_l} |\partial x_l u|^2 ds$$
$$\leq int_{\tilde{V}} |partial x_l u|^2$$

for $\tilde{V} = \{x + \epsilon s e_l : x \in supp(\varphi), s \in [0,1]\} \subset U$. Similarily,

$$\int_U |D_l^{-\epsilon}(\varphi^2 D_l^\epsilon u)|^2 \leq \int_U |D(\varphi^2 D_l^\epsilon u)|^2$$

It follows that:

$$\begin{aligned}\int \varphi^2 |D_l^\epsilon Du|^2 &\leq \delta \int_U |D(\varphi^2 D_l^\epsilon u)^2 + c_\delta \int_{\tilde{V}} |Du|^2 + |D_l^\epsilon u|^2 + u^2 + f^2 \\ &\leq c(\varphi^2 |D_l^\epsilon Du|^2 + |D_l^\epsilon u|^2)\end{aligned}$$

which gives for $\delta > 0$ sufficiently small, c depends on $max(|\varphi| + |D\varphi|)$ but not on u. We have:

$$\int_U \varphi^2 |D_l^\epsilon Du|^2 \leq c \int_{\tilde{V}} |Du|^2 + |u|^2 + |f|^2$$

Since $\varphi = 1$ on V, we obtain:

$$\int_V |D_l^\epsilon Du|^2 \leq \int_V \varphi^2 |D_l^\epsilon u Du|^2 \leq c \int_{\tilde{V}} |Du|^2 + |u|^2 + |f|^2$$

It follows that the sequence $(D_l^{1/\alpha} \partial x_i u)_\alpha$ is bounded in $L^2(V)$. Hence, using a result from Functional Analysis $(D_l^{1/\alpha} \partial x_i u)_\alpha$ converges up to a subsequence weakly in $L^2(V)$. By passing to the limit into $\forall v \in C_c^\infty(V)$,

$$\int_V \partial x_i u D_l^{-\epsilon} v = -\int_V v D_l^\epsilon \partial x_i u$$

We obtain that $\int_V \partial x_i u \partial x_l v = -\int_V v u_{il}$. Now we have:

$$\int_V |D \partial x_l u|^2 \leq \int_V |D_l^{1/\varphi(\alpha)} Du|^2 \leq c \int_{\tilde{V}} |Du|^2 + u^2 + f^2$$

It remains to show that $\int_{\tilde{V}} |Du|^2 \leq c \int_U u^2 + f^2$. Consider $v = \tilde{\varphi}^2 u$ where $\tilde{\varphi} \in C_c^\infty(U)$ is a cutoff function such that $\tilde{\varphi} = 1$ on $\tilde{V}$. Then:

$$\int_U \sum_{i,j}^n a_{ij} \partial x_i u \partial x_j (\tilde{\varphi(u)}) + (\sum_i b_i (\partial x_i u) + cu + f) \tilde{\varphi}^2 u = 0$$

Similarily, as before, we obtain that:

$$\int_{\tilde{V}} |Du|^2 \leq c \int_U u^2 + f^2$$

which proves case $k = 0$.

Case $k \geq 1$: We proceed by induction. Assume that the result holds $\forall k' \in [0, k-1]$. Then we write: $\forall \alpha$ such that $|\alpha| \leq k$, $\forall v \in C_c^\infty(U)$, since $D_v^\alpha \in C_c^\infty(U)$,

$$\int_U \sum_{i,j}^n a_{ij} \partial_{x_i} u \partial_{x_j} D^\alpha v + (\sum_i b_i \partial_{x_i} u + cu - f) D^\alpha v = 0$$

Integrating by parts, we obtain:

$$\int_U \sum_{i,j} a_{ij} \partial_{x_i} D^\alpha u \partial_{x_j} v + (\sum_i b_i \partial_{x_i} D^\alpha u + c D^\alpha u - \tilde{f}) v = 0$$

where $\tilde{f}$ depends on derivatives of a_{ij}, b_i, c, u, and f of order $\leq k$ and depends linearly on the derivatives of u and f.

$D^\alpha u$ is a weak solution of $LD^\alpha u = \tilde{f}$ in U. It follows that,

$$
\begin{aligned}
||D^\alpha u||_{H^2(V)} \leq c(||D^\alpha u||_{L^2(\tilde{V})}) + ||\tilde{f}||_{L^2(\tilde{V})} \\
&\leq c'(||u||_{H^k(U)} + ||f||_{H^{k-1}(U)}) \\
&\leq c''(||u||_{L^2(U)} + ||f||_{H^k(U)})
\end{aligned}
$$

for some constants c, c', c'' independent of u and f. This completes the proof. □

Remark 4.6. (i) It can be shown that if $f \in W^{k,p}(U)$, $p \in (1, \infty)$, then $u \in W^{k+2,p}_{loc}(U)$.

(ii) It can be shown that $W^{k,p}(V) \subset C^j(\overline{V})$ when $k > \frac{n}{p} + j$, V is open bounded and ∂V is C^1. Therefore if $a_{ij}, b_i, c, f \in C^\infty(U)$ then we obtain that $u \in C^\infty(U)$.

Theorem 4.11. (Global Regularity) Assume that U is bounded, ∂U is C^1, $a_{ij} \in C^{k+1}(\overline{U})$, $a_{ij} = a_{ji}$, $b_i, c \in C^k(\overline{U})$, $f \in C^k(\overline{U})$, $f \in H^k(U)$, $g \in H^{k+2}(U)$, for some $k \in \mathbb{N} \cup \{0\}$, L is uniformly elliptic in U, and u is a weak solution of: $\begin{cases} Lu = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$ Then $u \in H^{k+2}(U)$ and there exists $C > 0$ indepedent of u, f, g such that

$$||u||_{H^{k+2}(U)} \leq C(||u||_{L^2(U)} + ||f||_{H^k(U)} + ||g||_{H^{k+2}(U)})$$

Remark 4.7. (i) Can replace H^k and H^{k+2} by $W^{k,p}$ and $W^{k+2,p}$ respectively.

(ii) Combined with $W^{k,p} \subset C^j$ we obtain that if $a_{ij}, b_i, c, f, g \in C^\infty(\overline{U})$ then $u \in C^\infty(\overline{U})$.