Partial Differential Equations

Fall 2025, Math 475 Course Notes

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Contents

First-Order Linear PDEs

In this section, we will dive into some basic and elementary structures of partial differential equations. Throughout the section, consider $u(x_1, \cdots, x_n)$ as a multivariate function, $u_{x_1}, \cdots, u_{x_n}$ as its first order partial derivatives, and $u_{x_i x_j} \cdots$ as second order partial derivatives and so on.

1.1 Constant Coefficient Linear PDEs

In this part, let $u(x,y) : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, $a, b \in \mathbb{R}$ not both zero, and we are interested in the PDE of the form

$$\boxed{au_x + bu_y = 0} \tag{1.1}$$

A very intuitive example is $u(x,y) = bx - ay$, where $u_x = b, u_y = -a$ and $au_x + bu_y = ab - ab = 0$. But in general how can we solve it? We will give two methods.

1.1.1 Geometric Method Via Directional Derivative

When we have a multivariate function (in our example it suffices to illustrate with 2 variables) $z = u(x,y)$, the two partial derivatives are defined by

Definition 1. *Let $z = u(x,y)$ be a multivariate function, then the partial derivatives u_x, u_y at (x_0, y_0) are defined as*

$$u_x := \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} \tag{1.2}$$

$$u_y := \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}. \tag{1.3}$$

Just like normal derivative with one variable, the partial derivatives u_x, u_y are the rate of change along the x, y axis respectively. We may further extend this idea to any direction, say the "partial derivative" or the "rate of change" along the line $y = x$, $y = 2x - 3$ (a linear combination of x, y coordinates) etc. That's where we introduce directional derivatives. So we may then define the directional derivative along a vector $\mathbf{v} = (a, b)$:

Definition 2. *The directional derivative of $u(x,y)$ at (x_0, y_0) along $=(a,b)$ is given by*

$$D_{\mathbf{v}} u(x_0, y_0) = \lim_{h \to 0} \frac{u(x_0 + ha, y_0 + hb) - u(x_0, y_0)}{h} \tag{1.4}$$

and we can easily see that u_x, u_y is just the special case when $\mathbf{v} = (1, 0), (0, 1)$ respectively. Below is a figure to illustrate directional derivative geometrically:
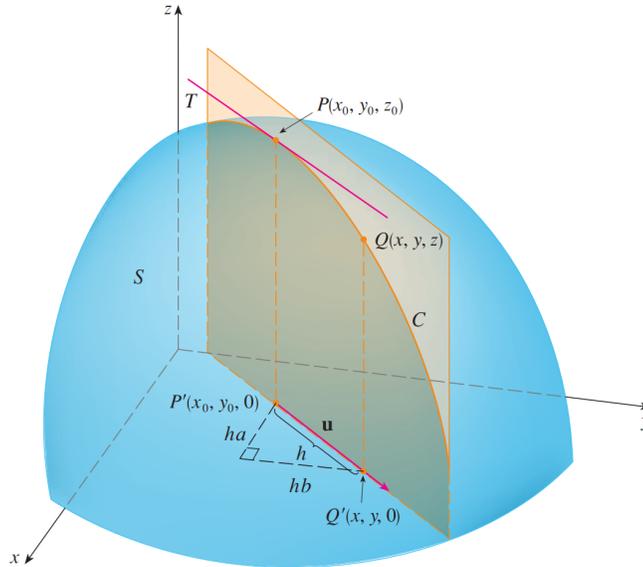
Figure 1: The directional derivative at (x_0, y_0, z_0) of the function $z = u(x, y)$. It can be viewed as the slope of tangent line of the curve obtained by slicing the function with a vertical plane that passes through the direction of the directional derivative. (*Credit: James-Stewart Calculus, early transcendentals, Eighth Edition, Page 946*)

Theorem 1. *The directional derivative of $u(x, y)$ along $\mathbf{v} = (a, b)$ can be computed by*

$$D_{\mathbf{v}}u(x, y) = u_x(x, y) \cdot a + u_y(x, y) \cdot b \tag{1.5}$$

The proof of this theorem can be done by using Chain rule and hence the readers are encouraged to try it themselves.

After knowing some directional derivatives, we go back to the equation $a \cdot u_x + b \cdot u_y = 0$, we realized this means the directional derivative of $u(x, y)$ along $\mathbf{v} = (a, b)$ is equal to zero, meaning $u(x, y)$ is constant along $\mathbf{v}$. The line equation of $\mathbf{v}$ can be expressed by $bx - ay = 0$, and the set of lines parallel to $\mathbf{v}$ has the general form of $bx - ay = C$, where $C \in \mathbb{R}$. They are called the *characteristic lines*, and on each of those lines $D_{\mathbf{v}}u(x, y)$ is a constant and the entire plane $\mathbb{R}^2$ is generated by the set of all characteristic lines. So we see that in this case $u(x, y)$ only depends on $bx - ay$, i.e which characteristic line it belongs to, and hence the general solution to $au_x + bu_y = 0$ is

$$\boxed{u(x, y) = f(bx - ay)} \tag{1.6}$$

where f is a function of one variable, and the exact f may be obtained once extra conditions are given. To verify the correctness of (1.6), we two partial derivatives:

$$u_x = bf'(bx - ay), u_y = -af'(bx - ay) \tag{1.7}$$

and hence $au_x + bu_y = abf'(bx - ay) - abf'(bx - ay) = 0$.

Example 1. *Solve the PDE $4u_x - 3u_y = 0$ with auxiliary condition $u(0, y) = y^3$.*

Solution: It is first easy to see that $u(x,y) = f(-3x - 4y)$ is the general solution, where $-3x - 4y = C$ is the characteristic line. Then plugging in the auxiliary condition, $y^3 = u(0,y) = f(-4y)$ we get $f(-4y) = y^3$ and by a change of variable we see that $f(t) = -t^3/64$. So we have $u(x,y) = f(-3x - 4y) = (-3x - 4y)^3/64$.

Example 2. *Solve the PDE $xu_x + yu_y = 0$.*

We notice that the directional derivatives along (x,y) is constant zero, and we have

$$\frac{dy}{dx} = \frac{y}{x} \tag{1.8}$$

which brings us to a separable differential equation and we have

$$\int \frac{1}{y} dy = \int \frac{1}{x} dx + C \tag{1.9}$$

and we have $\ln|y| - \ln|x| = C$, which is the characteristic curve of the PDE, and given any function g defined through $g(\ln|y| - \ln|x|)$ will be a solution.

1.1.2 Change of Variable Method

Now we present a different method using change of variables. Imagine using the direction of $\mathbf{v} = (a,b)$ as our new coordinates x' and $\mathbf{v}^\perp$ as y'. Then we can verify that

$$x' = ax + by \qquad y' = bx - ay \tag{1.10}$$

and hence we have the relation that

$$x = \frac{ax' + by'}{a^2 + b^2} \qquad y = \frac{bx' - ay'}{a^2 + b^2} \tag{1.11}$$

and by Chain rule, we get

$$u_x = \frac{\partial u}{\partial x'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'} \cdot \frac{\partial y'}{\partial x} = au_{x'} + bu_{y'} \tag{1.12}$$

$$u_y = \frac{\partial u}{\partial x'} \cdot \frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'} \cdot \frac{\partial y'}{\partial y} = bu_{x'} - au_{y'} \tag{1.13}$$

so we have

$$au_x + bu_y = a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = (a^2 + b^2)u_{x'} \tag{1.14}$$

and we obtain a new equation $(a^2 + b^2)u_{x'} = 0$, which indicates the solution $u(x,y)$ is independent of x' and hence we have $u(x',y') = f(y') = f(bx - ay)$. Note that this is the exact same answer we get using the directional derivative method.

Then, we are interested in a more general form

$$\boxed{au_x + bu_y = f(x,y)} \tag{1.15}$$

for a given function $f(x,y)$. If f reduces to zero then it is the case in section 1.1.1.

Theorem 2. *The solution to (1.13) is given by*

$$u(x,y) = \frac{1}{\sqrt{a^2+b^2}} \int_L f\,ds + g(bx - ay) \tag{1.16}$$

where L is the characteristic line segment from y axis to (x,y).

Proof. We will again use the change of variable method, using the change of variables discussed in (1.8), and the relations stated in (1.9);(1.10);(1.11), we have

$$a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = f\left(\frac{ax' + by'}{a^2+b^2}, \frac{bx' - ay'}{a^2+b^2}\right) \tag{1.17}$$

which simplifies to

$$u_{x'} = \frac{1}{a^2+b^2} \cdot f\left(\frac{ax' + by'}{a^2+b^2}, \frac{bx' - ay'}{a^2+b^2}\right) \tag{1.18}$$

hence integrate 1.16 with respect to x' gives

$$u(x',y') = \int_{x'} \frac{1}{a^2+b^2} \cdot f\left(\frac{ar + by'}{a^2+b^2}, \frac{br - ay'}{a^2+b^2}\right) dr + g(y') \tag{1.19}$$

and we now use the original coordinates and we have

$$u(x,y) = \int_{ax+by} \frac{1}{a^2+b^2} \cdot f\left(\frac{ar + b(bx - ay)}{a^2+b^2}, \frac{br - a(bx - ay)}{a^2+b^2}\right) dr + g(bx - ay) \tag{1.20}$$

by a simple change of variable $s = \dfrac{r}{\sqrt{a^2+b^2}}$, we have

$$u(x,y) = \frac{1}{\sqrt{a^2+b^2}} \int_{\frac{ax+by}{\sqrt{a^2+b^2}}} f\left(\frac{b^2x + a(\sqrt{a^2+b^2}s - by)}{a^2+b^2}, \frac{a^2y + b(\sqrt{a^2+b^2}s - ax)}{a^2+b^2}\right) ds + g(bx - ay) \tag{1.21}$$

and denoted by

$$u(x,y) = \frac{1}{\sqrt{a^2+b^2}} \int_L f\,ds + g(bx - ay) \tag{1.22}$$

and as we can see L is the line segment from y axis to the point (x,y).

∎

Example: Solve the PDE $u_x + u_y = 1$.

Solution: We follow the result of theorem 2, we have $f(x,y) = 1, a = b = 1$, then we plug these values into (1.20), we have

$$u(x,y) = \frac{1}{\sqrt{2}} \int_{\frac{x+y}{\sqrt{2}}} ds + g(x - y)$$

$$= \frac{x+y}{2} + g(x - y)$$

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and we get

$$u_x = \frac{1}{2} + g'(x-y) \qquad u_y = \frac{1}{2} - g'(x-y) \tag{1.23}$$

where $g(x-y)$ is an arbitrary single variable function that depends through $x-y$, and notice that $u_x + u_y = 1$ which means the solution is valid.

Example: Solve the PDE $u_x + u_y = xy$.
Solution: In this example, we have $f(x,y) = xy, a = b = 1$ and (1.20) gives us

$$u(x,y) = \frac{1}{\sqrt{2}} \int_{\frac{x+y}{\sqrt{2}}} \left(\frac{x + \sqrt{2}s - y}{2} \right) \cdot \left(\frac{y + \sqrt{2}s - x}{2} \right) ds + g(x-y) \tag{1.24}$$

$$= \frac{1}{\sqrt{2}} \int_{\frac{x+y}{\sqrt{2}}} \frac{2s^2 - (x-y)^2}{4} ds + g(x,y) \tag{1.25}$$

$$= \left[\frac{2s^3 - 3s(x-y)^2}{12} \right] \Bigg|_{s = \frac{x+y}{\sqrt{2}}} + g(x-y) \tag{1.26}$$

$$= -\frac{(x+y)(x^2 + y^2 - 4xy)}{12} + g(x-y) \tag{1.27}$$

hence we get the general solution to the original PDE:

$$u(x,y) = -\frac{(x+y)(x^2 + y^2 - 4xy)}{12} + g(x-y) \tag{1.28}$$

where $g(x-y)$ is an arbitrary single valued function which depends through $x-y$. To verify, note that we have

$$u_x = -\frac{1}{12}\left(x^2 + y^2 - 4xy + (x+y)(2x-4y)\right) + g'(x-y)$$
$$= \frac{1}{4}\left(-x^2 + y^2 + 2xy\right) + g'(x-y) \tag{1.29}$$

and similarly

$$u_y = -\frac{1}{12}\left(x^2 + y^2 - 4xy + (x+y)(2y-4x)\right) - g'(x-y)$$
$$= \frac{1}{4}\left(x^2 - y^2 + 2xy\right) - g'(x-y) \tag{1.30}$$

where

$$u_x + u_y = \frac{1}{4}\left(-x^2 + y^2 + 2xy + x^2 - y^2 + 2xy\right) + g'(x-y) - g'(x-y)$$
$$= xy. \tag{1.31}$$

1.2 Nonhomogeneous Problem and Method of Characteristics

We now generalize the idea of transport equation into $\mathbb{R}^n$. Suppose $x, b \in \mathbb{R}^n$, $t \in (0, \infty)$ and Du is the gradient of u with respect to x_i, then consider the problem

$$\begin{cases} u_t + b \cdot Du = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{1.32}$$

To solve this, we inspired by the homogeneous problem, suppose we set $z(s) := u(x + sb, t + s)$ for $s \in \mathbb{R}$, then differentiating z will yield

$$\dot{z}(s) = b \cdot Du(x + sb, t + s) + u_t(x + sb, t + s) = f(x + sb, t + s) \tag{1.33}$$

then we notice that $u(x, t) = u(x + 0b, t + 0) = z(0)$ and $u(x - tb, t - s) = u(x - tb, 0) = g(x - tb) = z(-t)$, so we have

$$u(x, t) - g(x - tb) = z(0) - z(-t) = \int_{-t}^{0} \dot{z}(s) ds \tag{1.34}$$

$$= \int_{-t}^{0} f(x + sb, t + s) ds \tag{1.35}$$

$$= \int_{0}^{t} f(x + (s - t)b, s) ds \tag{1.36}$$

hence, we have the general solution given by

$$u(x, t) = g(x - tb) + \int_{0}^{t} f(x + (s - t)b, s) ds \tag{1.37}$$

Example 3. *Solve the PDE $u_t + u_x = 1$ with $u(x, 0) = x^2$.*

In this case observe that $f \equiv 1$, $g = x^2$, $Du = u_x$, $b = 1$, so by the formula we proposed, we have

$$u(x, t) = (x - t)^2 + \int_{0}^{t} 1 ds = x^2 - 2xt + t^2 + t. \tag{1.38}$$

It is easy to see that the function above solves the PDE.

We now study a more general form: $F(Du, u, x) = 0$ in U subject to the boundary condition $u = g$ on Γ where $\Gamma \subseteq \partial U$, $g : \Gamma \to \mathbb{R}$, also $F, g \in C^\infty$. So what is the *method of characteristics*? Well we pick $x \in U$, and then another $x_0 \in \Gamma$. Since we already know on Γ, $u = g$ so we are able to compute the value at x_0. Then we try to connect x_0, x by some nice curve u that solves the PDE.

Suppose such a curve which is described parametrically by $\mathbf{x}(s) = (x^1(s), \cdots, x^n(s))$ where $s \in \mathbb{R}$ is our parameter. We assume $u \in C^2$ solves the PDE $F(Du, u, x) = 0$ and let $z(s) = u(\mathbf{x}(s))$, as well as $\mathbf{p}(s) := Du(\mathbf{x}(s))$ where $p^i(s) = u_{x_i}(\mathbf{x}(s))$. In our set up, z gives the values along the curve, $\mathbf{p}$ gives the values of the gradient Du. Note that

$$\dot{p}^i(s) := \frac{d}{ds} p^i(s) = \frac{d}{ds} \sum_{j=1}^{n} u_{x_i x_j}(\mathbf{x}(s)) \dot{x}^j(s) \tag{1.39}$$

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also we differentiate $F(Du, u, x) = 0$ with respect to x^i:

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(Du, u, x) u_{x_j x_i} + \frac{\partial F}{\partial z}(Du, u, x) u_{x_i} + \frac{\partial F}{\partial x_i}(Du, u, x) = 0 \qquad (1.40)$$

set

$$\dot{x}^j(s) = \frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)) \qquad (1.41)$$

then we evaluate (1.40) at $x = \mathbf{x}(s)$, we have

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)) u_{x_i x_j}(\mathbf{x}(s)) + \frac{\partial F}{\partial z}(\mathbf{p}(s), z(s), \mathbf{x}(s)) p^i(s) + \frac{\partial F}{\partial x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) = 0 \qquad (1.42)$$

so now use the expression above and (1.41), substitute back into (1,39) will give us

$$\dot{p}^i(s) = -\frac{\partial F}{\partial x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) - \frac{\partial F}{\partial z}(\mathbf{p}(s), z(s), \mathbf{x}(s)) p^i(s) \qquad (1.43)$$

and finally differentiating $z = u(\mathbf{x}(s))$ will yield

$$\dot{z}(s) = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(\mathbf{x}(s)) \dot{x}^j(s) = \sum_{j=1}^{n} p^j(s) \frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)). \qquad (1.44)$$

Definition 3. *We define the following systems of ODEs as the characteristic equations of the PDE $F(Du, u, x) = 0$:*

$$\begin{cases} \dot{\mathbf{p}}(s) & = -D_x F(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_z F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \mathbf{p}(s) \\ \dot{z}(s) & = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \cdot \mathbf{p}(s) \\ \dot{\mathbf{x}}(s) & = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \end{cases} \qquad (1.45)$$

Expression (1.45) looks a little bit cursed, but let us consider a simple example:

$$a(x, y) u_x + b(x, y) u_y = c(x, y) u + d(x, y) \qquad (1.46)$$

for function $u(x, y)$. Then the characteristic curve will be

$$\begin{cases} \dot{x}(s) = a(x(s), y(s)) \\ \dot{y}(s) = b(x(s), y(s)) \\ \dot{z}(s) = c(x(s), y(s)) z + d(x(s), y(s)) \end{cases} \qquad (1.47)$$

Example 4. *Solve the PDE $-y u_x + x u_y = u$ with boundary condition $u(x, 0) = g(x)$.*

It is easy to see that $\dot{x}(s) = -y, \dot{y}(s) = x, \dot{z}(s) = z$. We first solve the system of ODEs with x, y:

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases} \implies \begin{cases} x = c_2 \cos(s) - c_1 \sin(s) \\ y = c_1 \cos(s) + c_2 \sin(s) \end{cases} \qquad (1.48)$$

Let $x(0) = x_0, y(0) = 0$, then $x(s) = x_0 \cos(s), y(s) = x_0 \sin(s)$. Furthermore solve for z we have $z(s) = c_3 e^s$. With initial condition, $z(0) = g(x_0)$ so we have $z(s) = g(x_0) e^s$. Now given (x, y) we need to solve for x_0 and s such that they passes through the characteristics, we solve $x = x_0 \cos(s), y = x_0 \sin(s)$ and hence $x_0^2 = x^2 + y^2, s = \arctan(y/x)$ and hence we obtain

$$u(x, y) = g(x_0) e^s = g(\sqrt{x^2 + y^2}) e^{\arctan(y/x)}. \qquad (1.49)$$

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Example 5. *Solve the PDE $u_x + u_y + u = e^{x+2y}$ with boundary condition $u(x,0) = 0$.*

We first note that $\dot{x}(s) = 1, \dot{y}(s) = 1$ and $\dot{z}(s) = -1 + e^{x+2y}$, first solve the system of ODEs with x, y, we obtain

$$\begin{cases} \dot{x}(s) = 1 \\ \dot{y}(s) = 1 \end{cases} \implies \begin{cases} x(s) = s + c_1 \\ y(s) = s + c_2 \end{cases} \tag{1.50}$$

for some constant $c_1, c_2 \in \mathbb{R}$. We now plug in the initial condition, we know that $x(0) = x_0, y(0) = 0$ (we make it this way so the point lie on the "boundary" where we know $u = g$), hence we have

$$x(s) = s + x_0 \qquad y(s) = s \tag{1.51}$$

with this information, we substitute x, y by s into z and have $\dot{z}(s) = -z(s) + e^{3s+x_0}$, which is an ODE that can be solved by using integrating factors. Note that

$$\mu(s) = \exp\left(\int 1 ds\right) \qquad z(s) = \frac{1}{\mu(s)}\left(\int \mu(s) e^{3s+x_0} ds + C\right) \tag{1.52}$$

and we hence have

$$z(s) = \frac{1}{4} e^{x_0 + 3s} + \frac{C}{e^s}, C \in \mathbb{R} \tag{1.53}$$

with initial condition $z(0) = 0$, we have

$$z(s) = \frac{1}{4} e^{x_0 + 3s} - \frac{1}{4} e^{x_0 - s} \tag{1.54}$$

we then use (1.51) to obtain x_0, s in terms of x, y, where $s = y, x_0 = x - y$, hence we have

$$u(x,y) = \frac{1}{4}(e^{x+2y} - e^{x-2y}). \tag{1.55}$$

The above two examples motivate us to further investigate (1.45) and derive a more specific formula for different type of PDEs:

Linear PDEs: The first case we consider is when $F(Du, u, x)$ is linear and homogeneous, which means it takes the form

$$b(x(s)) \cdot Du + c(x)u = 0 \tag{1.56}$$

and it suggests that we have $F(p, z, x) = b(x(s)) \cdot p(s) + c(x(s))z(s)$, where we have $D_p F = b(x)$ and then by (1.45)

$$\dot{x}(s) = D_p F = b(x(s)) \qquad \dot{z}(s) = D_p F \cdot p(s) = b(x(s)) \cdot p(s) \tag{1.57}$$

also note that we have $F(p, x, z) = b(x) \cdot p(s) + c(x)z(s) = 0$ hence we further have the expression

$$\dot{z}(s) = -c(x(s))z(s) \tag{1.58}$$

so in summary we have

$$\begin{cases} \dot{x}(s) = b(x(s)) \\ \dot{z}(s) = -c(x(s))z(s) \end{cases} \tag{1.59}$$

as our parametrization and from this point we can easily see the parametrization we get in example 4 and 5.

Quasi-linear PDEs: We now consider the PDE of the form

$$F(Du, u, x) = b(x, u(x)) \cdot Du + c(x, u(x)) = 0 \tag{1.60}$$

and we now have the parametrization $F(p, x, z) = b(x, z) \cdot p + c(x, z) = 0$, and from here we have $D_p F = b(x(s), z(s))$ so $\dot{x}(s) = b(x(s), z(s))$, and $\dot{z}(s) = b(x(s), z(s)) \cdot p(s)$. We finally use the fact that $b(x, z) \cdot p + c(x, z) = 0$ to conclude $\dot{z}(s) = -c(x(s), z(s))$ so im summary we have

$$\begin{cases} \dot{x}(s) = b(x(s), z(s)) \\ \dot{z}(s) = -c(x(s), z(s)) \end{cases} . \tag{1.61}$$

Example 6. *Solve the PDE $u_x + u_y = u^2$ with boundary condition $u(x, 0) = g$.*

We first see that $\dot{x} = 1, \dot{y} = 1$ so we have the solution

$$\begin{cases} x(s) = s + c_1 \\ y(s) = s + c_2 \end{cases} \tag{1.62}$$

with the boundary condition, we let x_0 be the point on g such that our parametrization yields $x(0) = x_0, y(0) = 0$ and hence we have

$$\begin{cases} x(s) = s + x_0 \\ y(s) = s \end{cases} . \tag{1.63}$$

Now for z we have $\dot{z} = z^2$, which now turns into a separable ODE and we have

$$\int \frac{1}{z^2} dz = \int 1 ds + C \implies -\frac{1}{z} = s + C \tag{1.64}$$

take $s = 0$, and let $g(x_0) = z_0$, then we have $-\frac{1}{z_0} = C$ hence then we solve for z and we get

$$z(s) = \frac{z_0}{1 - sz_0} := \frac{g(x_0)}{1 - sg(x_0)} \tag{1.65}$$

Note that in (1.63) our parametrization will yield $s = y; x_0 = x - y$ so we substitute into (1.65) and we get

$$u(x, y) = \frac{g(x - y)}{1 - y \cdot g(x - y)} \tag{1.66}$$

as the solution to the PDE, where g is any function depend through $x - y$. But notice that the solution only makes sense only $1 - yg(x - y) \neq 0$.

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Wave Equations

2.1 Interpretations and Derivations

Suppose $U \subseteq \mathbb{R}^n$ is open, and we define a function $u(x,t) : \bar{U} \times [0,\infty) \to \mathbb{R}$ where $x \in U$ and t is the time variable.

Definition 4. *The wave equation is defined by*

$$u_{tt} - \Delta u = 0 \tag{2.1}$$

where Δ is the Laplacian operator.

We first provide some physical interpretations to the wave equation. Suppose we have a smooth subregion $V \subseteq U$, and we consider the acceleration within V. The interpretation is easier for $n = 1, 2$ or 3. Let $u(x,t)$ be the displacement in some direction of the point x at time $t \geq 0$, and the acceleration within V is then

$$\frac{d^2}{dt^2} \int_V u \, dx = \int_V u_{tt} \, dx \tag{2.2}$$

while the net force defined on ∂V is given by

$$-\int_{\partial V} \mathbf{F} \cdot \nu \, dS \tag{2.3}$$

where $\mathbf{F}$ is the force acting on V through ∂V, ν is the outward normal unit vector, then by Newton's second law,

$$\int_V u_{tt} \, dx = -\int_{\partial V} \mathbf{F} \cdot \nu \, dS \tag{2.4}$$

where we take the mass of the object to be unit mass, and further this identity obtains for all subregion V so we further have $u_{tt} = -\mathrm{div}\mathbf{F}$. Further for elastic bodies $\mathbf{F}$ is a function of the displacement gradient Du, hence we have the system

$$u_{tt} = \mathrm{div}\mathbf{F}(Du) = 0 \tag{2.5}$$

and for small vibrations we use the approximation formula $\mathbf{F}(Du) = -aDu$ so we have $u_{tt} - a\Delta u = 0$ and wave equation is the case when $a = 1$.

2.2 d'Alembert's Formula

We first focus on the initial value problem for $u(x,t)$ defined on $\mathbb{R} \times (0,\infty)$. Consider the system

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{2.6}$$

Theorem 3. *The general solution to the wave equation $u_{tt} = c^2 u_{xx}$ takes the form*

$$u(x,t) = f(x+ct) + g(x-ct) \tag{2.7}$$

through some functions f, g.

Proof. By direct computation, we have

$$u_x = f'(x+ct) + g'(x-ct), u_{xx} = f''(x+ct) + g''(x-ct) \tag{2.8}$$

and

$$u_t = f'(x+ct) \cdot c + g'(x-ct) \cdot (-c), u_{tt} = c^2 f''(x+ct) + c^2 g''(x-ct) \tag{2.9}$$

and the results follows trivially. ∎

Theorem 4. *The d'Alembert's formula is given by*

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy \quad for \ x \in \mathbb{R}, t \geq 0. \tag{2.10}$$

with initial condition $u(x,0) = g, u_x(x,0) = h$

Proof. (Evan's) Observe that we can rewrite the PDE as

$$u_{tt} - u_{xx} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u = 0 \tag{2.11}$$

we define v as

$$v(x,t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u(x,t) \tag{2.12}$$

then we have $v_t(x,t) + v_x(x,t) = 0$. This is a transport equation, where we see the directional derivative along $(1,1)$ is zero and the function is constant along this constraint. So we have a general solution defined by $v(x,t) = a(x-t)$ for any function a that defined through $x-t$ and $a(x) = v(x,0)$. Hence we finally have

$$u_t(x,t) - u_x(x,t) = a(x-t) \qquad \text{in } \mathbb{R} \times (0,\infty) \tag{2.13}$$

which appears to be another transport equation, the general solution takes the form

$$u(x,t) = \int_0^t a(x+(t-s)-s)ds + b(x+t) \tag{2.14}$$

$$= \frac{1}{2}\int_{x-t}^{x+t} a(y)dy + b(x+t) \tag{2.15}$$

with $b(x) := b(x,0)$. With initial conditions $b(x) = g(x)$ when $t = 0$, also

$$a(x) = v(x,0) = u_t(x,0) - u_x(x,0) = h(x) - g'(x) \tag{2.16}$$

we finally have

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t}\left(g(y) - g'(y)\right)dy + g(x+t). \tag{2.17}$$

after simplification we have

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy, x \in \mathbb{R}, t \geq 0. \tag{2.18}$$

∎

And d'Alembert's formula will give us the solution of the PDE equipped with the boundary value problem. If in general we have $u_{tt} - c^2 u_{xx} = 0$ for some $c \in \mathbb{R}$, then d'Alenbert's formula will yield

$$u(x,t) = \frac{1}{2}[g(x+ct) + g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy \tag{2.19}$$

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2.3 Energy Conservation

We go back to the 1D wave equation about a vibrating string:

$$u_{tt} - c^2 u_{xx} = 0 \tag{2.20}$$

We first try to derive this formula physically (not via real math). Consider we have an infinity long string with two boundaries fixed, and now we apply a vibration to the string so it will move vertically.

Figure 2: A vibrating string *(Source: Strauss PDE)*

At some moment, the string will look like figure 2. Let x to be a point on the string indicating its position, we denote $u(x,t)$ as the displacement from equilibrium position at time t. If we pick two points $x_0 < x_1$ close enough, and we zoom in into the string and then that part of the string will look like figure 3:
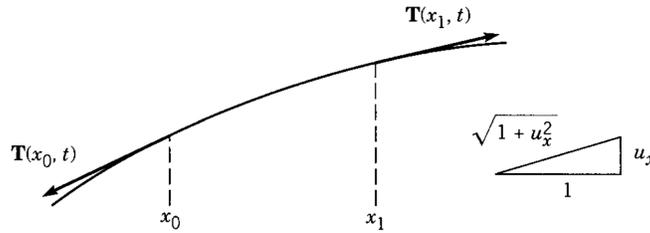

Figure 3: A zoomed vibrating string *(Source: Strauss PDE)*

We assume the string is flexible, elastic and homogeneous with density ρ, so at each point its tension vector is tangential to the string, and we further assume $\mathbf{T}$ does not depend on time t. We use $\mathbf{T}(x_1), \mathbf{T}(x_0)$ to denote the magnitude of the tension vector for the points x_0, x_1 at time t. At point a point x, u_x can be viewed as the tangent line of the string at this point, and the small triangle in figure 2 illustrates this. Since $\mathbf{T}(x)$ is parallel to the tangent line, we can then decompose the tension vector $\mathbf{T}_x$ into two directions $\mathbf{T}_x$ and $\mathbf{T}_y$. Since we assumed the string only moves vertical, so $\mathbf{T}_x = 0$. Then we have

$$\mathbf{T}_y = \mathbf{T}(x)\sin\theta = \mathbf{T}(x) \cdot \frac{u_x}{\sqrt{1+u_x^2}} \tag{2.21}$$

In segment x_0 to x_1, the net tension is simply the tension on its boundary since inside tension will cancel out, given by $\mathbf{T}(x)\sin\theta\Big|_{x=x_0}^{x=x_1}$, and at each point its mass times acceleration is just ρu_{tt}, then by integrating over the region, and apply Newton's law, we have

$$\frac{\mathbf{T}(x)u_x}{\sqrt{1+u_x^2}}\Bigg|_{x=x_0}^{x=x_1} = \int_{x_0}^{x_1} \rho u_{tt}dx \tag{2.22}$$

when the vibration is small, we immerse the philosophy from the holy engineers and write $\sqrt{1+u_x^2} = 1$, so we have

$$\mathbf{T}(x_1)u_{x_1} - \mathbf{T}(x_0)u_{x_0} = \int_{x_0}^{x_1} \rho u_{tt}dx \tag{2.23}$$

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if we also assume $\mathbf{T}$ does not depend on x, differentiating the term above with respect to x will give us

$$(\mathbf{T}u_x)_x = \rho u_{tt} \tag{2.24}$$

where we let $c = \sqrt{\dfrac{\mathbf{T}}{\rho}}$ and hence we have

$$\boxed{u_{tt} = c^2 u_{xx}} \tag{2.25}$$

as the wave equation in one dimension.

We now talk about energy conservation in wave equation. Again for simplicity we first consider one-dimensional case.

Definition 5. *Let $u_{tt} = c^2 u_{xx}$ be the wave equation of a vibrating string, we define the kinetic energy KE and potential energy PE of the string by the following:*

$$KE = \frac{1}{2}\rho \int u_t^2 dx \qquad PE = \frac{1}{2}\mathbf{T}\int u_x^2 dx \tag{2.26}$$

Theorem 5. *In wave equation the total energy $E = KE + PE$ is conserved.*

Proof. We first differentiate the kinetic energy:

$$\frac{d}{dt}KE = \frac{d}{dt}\frac{1}{2}\rho\int u_t^2 dx = \frac{1}{2}\rho\int \frac{d}{dt}u_t^2 dx = \rho\int u_t u_{tt} dx \tag{2.27}$$

then we substitute u_{tt} with $c^2 u_{xx}$ and perform an integration by part:

$$\frac{d}{dt}KE = \rho c^2 \int u_t u_{xx} dx = \mathbf{T}\left(u_t u_x\right)\Big|_{x=-\infty}^{x=\infty} - \mathbf{T}\int u_x u_{tx} dx \tag{2.28}$$

where after simplifications we have

$$\frac{d}{dt}KE = -\mathbf{T}\int u_x u_{tx} dx \tag{2.29}$$

finally observe that

$$\frac{d}{dt}PE = \frac{d}{dt}\frac{1}{2}\mathbf{T}\int u_x^2 dx = \mathbf{T}\int u_x u_{xt} dx \tag{2.30}$$

it means we have

$$\frac{d}{dt}(KE + PE) = 0 \tag{2.31}$$

and hence the total energy is conserved.

$\blacksquare$

Dirac Delta Function and Distributions

3.1   Basic Definitions and Properties

Definition 6. *A distribution F is an operator $F : \varphi \in C_C^\infty(\mathbb{R}) \to \mathbb{R}$ that is linear and continuous.*

For linearity, it means for all $\varphi_1, \varphi_2 \in C_C^\infty(\mathbb{R})$ and $c \in \mathbb{R}$, $F(\varphi_1 + c\varphi_2) = F(\varphi_1) + cF(\varphi_2)$; For continuous is means for all sequence $\{\varphi_n\} \subseteq C_c^\infty(\mathbb{R})$ and $\varphi \in C_c^\infty(\mathbb{R})$, if $\varphi_n \to \varphi$ in $C_C^\infty(\mathbb{R})$ then $F(\varphi_n) \to F(\varphi)$. *Note that convergence in $C_C^\infty(\mathbb{R})$ means $\varphi_n^{(k)} \to \varphi^{(k)}$ uniformly for all derivatives $k = 0, 1, \cdots$, and uniformly convergence means $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\sup ||\varphi_n^{(k)}(x) - \varphi^{(k)}(x)|| < \varepsilon$ for all x and $n \geq N$.*

For notation, we write $\langle F, \varphi \rangle$ for distributions for the action of the distribution F on $\varphi \in C_C^\infty(\mathbb{R})$, and then we can write the linearity as

$$\langle F, \varphi_1 + c\varphi_2 \rangle = \langle F, \varphi_1 \rangle + c \langle F, \varphi_2 \rangle \tag{3.1}$$

Definition 7. *Let $f(x)$ be a locally integrable function on $\mathbb{R}$, we define $F_f : C_C^\infty(\mathbb{R}) \to \mathbb{R}$ by*

$$\langle F_f, \varphi \rangle := \int_{\mathbb{R}} f(x)\varphi(x)dx \tag{3.2}$$

for all $\varphi(x) \in C_C^\infty(\mathbb{R})$.

We claim that (3.2) indeed defines a distribution, we can verify its linearity and continuity easily.

Definition 8. *Define δ_0 as the distribution $\langle \delta_0, \varphi \rangle = \varphi(0)$ for any $\varphi \in C_C^\infty(\mathbb{R})$.*

Example 7. *Consider the Heaviside step function $h(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$, then*

$$\langle F_h, \varphi \rangle = \int_{\mathbb{R}} h(x)\varphi(x)dx = \int_{\mathbb{R}^+} \varphi(x)dx \tag{3.3}$$

is a distribution.

We now consider the derivative of a distribution, for a function $f(x)$ and its derivative $f'(x)$, we can define the following distributions:

$$\langle F_f, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x)dx \qquad \langle F_{f'}, \varphi \rangle = \int_{\mathbb{R}} f'(x)\varphi(x)dx \tag{3.4}$$

using integration by parts, we have

$$\int_{\mathbb{R}} f'(x)\varphi(x)dx = f(x)\varphi(x)\Big|_{x=-\infty}^{x=\infty} - \int_{\mathbb{R}} f(x)\varphi'(x)dx = -\int_{\mathbb{R}} f(x)\varphi'(x)dx \tag{3.5}$$

because φ is compactly supported so $\varphi(\infty) = \varphi(-\infty) = 0$. Note that the last term in (4.5) is simply $-\langle F_f, \varphi' \rangle$, so we have the following relation:

$$\langle F_{f'}, \varphi \rangle = -\langle F_f, \varphi' \rangle \tag{3.6}$$

which also motivates the definition of the derivative of a distribution:

Definition 9. *Let F be a distribution, then F' is another distribution defined by*

$$\langle F', \varphi \rangle := - \langle F, \varphi' \rangle \tag{3.7}$$

for all $\varphi \in C_C^\infty(\mathbb{R})$.

Followed by this definition, we can compute the derivative of Dirac delta function:

$$\langle \delta_0', \varphi \rangle = - \langle \delta_0, \varphi' \rangle = - \int_{\mathbb{R}} \delta_0 \varphi(x) dx := -\varphi(0). \tag{3.8}$$

Example 8. *Consider the function $p(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$, then its distribution is given by*

$$\langle F_p, \varphi \rangle = \int_{\mathbb{R}} p(x)\varphi(x) dx = \int_{\mathbb{R}^+} x\varphi(x) dx \tag{3.9}$$

then consider $F_{p'}$, the derivative of such distribution, we have

$$\langle F_{p'}, \varphi \rangle = - \langle F_p, \varphi \rangle = - \int_{\mathbb{R}^+} x\varphi'(x) dx. \tag{3.10}$$

Then recall the distribution of the Heaviside step function:

$$\langle F_h, \varphi \rangle = \int_{\mathbb{R}^+} \varphi(x) dx := x\varphi(x) \Big|_{x=0}^{x=\infty} - \int_{\mathbb{R}^+} x\varphi'(x) dx = - \int_{\mathbb{R}^+} x\varphi'(x) dx \tag{3.11}$$

using integration by parts and the fact that $\varphi(\infty) = 0$. Then compare (3.10) and (3.9) we see that

$$\langle F_{p'}, \varphi \rangle = \langle F_h, \varphi \rangle. \tag{3.12}$$

Further, what will happen if we consider the derivative of the Heaviside distribution, F_h'? Let's see:

$$\langle F_h, \varphi \rangle = \langle F_H', \varphi \rangle = - \langle F_H, \varphi' \rangle = - \int_{\mathbb{R}^+} 1 \cdot \varphi'(x) dx = \varphi(0) \tag{3.13}$$

since again $\varphi(\infty) = 0$. Then by definition, we have

$$\langle F_h, \varphi \rangle = \langle \delta_0, \varphi \rangle. \tag{3.14}$$

That is, the derivative of the Heaviside function in the sense of distributions is the Dirac delta function.

Example 9. *Let $f(x) = |x|$, which is, non-differentiable at $x = 0$.*

We will try differentiations in the sense of distributions:

$$\langle F_f', \varphi \rangle = - \langle F_f, \varphi \rangle = \int_{\mathbb{R}} |x|\varphi'(x) dx = - \left(\int_{-\infty}^0 -x\varphi'(x) dx + \int_0^\infty x\varphi'(x) dx \right) \tag{3.15}$$

which is,

$$\langle F_f', \varphi \rangle = x\varphi(x) \Big|_{x=-\infty}^0 - \int_{-\infty}^0 \varphi(x) dx - x\varphi(x) \Big|_{x=0}^{x=\infty} + \int_0^\infty \varphi(x) dx \tag{3.16}$$

and after simplifications we have

$$\langle F'_f, \varphi \rangle = \int_{\mathbb{R}} g(x)\varphi(x)dx = \langle F_g, \varphi \rangle \tag{3.17}$$

where $g(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$.

Example 10. *Let* $f(x) = \begin{cases} x^2 & x \geq 0 \\ 2x+3 & x < 0 \end{cases}$, *then what is* f' *in the sense of distributions?*

Let us compute directly from the definition:

$$\langle F'_f, \varphi \rangle = - \langle F_f, \varphi' \rangle \tag{3.18}$$

$$= - \int_{-\infty}^{0} (2x+3)\varphi'(x)dx - \int_{0}^{\infty} x^2\varphi'(x)dx \tag{3.19}$$

$$= - \left((2x+3)\varphi(x) \Big|_{x=-\infty}^{0} - \int_{-\infty}^{0} 2\varphi(x)dx + x^2\varphi(x) \Big|_{x=0}^{x=\infty} - \int_{0}^{\infty} 2x\varphi(x)dx \right) \tag{3.20}$$

$$= - \left(3\varphi(0) - 2\int_{-\infty}^{0} \varphi(x)dx - \int_{0}^{\infty} 2x\varphi(x)dx \right) \tag{3.21}$$

$$= \int_{-\infty}^{\infty} g(x)\varphi(x)dx - 3\varphi(0) \tag{3.22}$$

$$:= \langle g, \varphi \rangle - 3\varphi(0) \tag{3.23}$$

$$= \langle g - 3\delta_0, \varphi \rangle. \tag{3.24}$$

where $g(x) = \begin{cases} 2 & x < 0 \\ 2x & x \geq 0 \end{cases}$. We say that the derivative of f in the sense of distributions is the distribution $g - 3\delta_0$, or $F_g - 3\delta_0$. The $3\delta_0$ is a result of the jump discontinuity in f at $x = 0$.

Let us consider a more general case of discontinuity. Define

$$f(x) = \begin{cases} h(x) & x < 0 \\ g(x) & x > 0 \end{cases} \tag{3.25}$$

where $g(0) - h(0) := c$ has a jump hence $f(x)$ is discontinuous at 0. We now consider the derivative of f in sense of distributions. By direct computation, we have

$$\langle F'_f, \varphi \rangle = - \langle F_f, \varphi' \rangle \tag{3.26}$$

$$= - \left(\int_{-\infty}^{0} h(x)\varphi'(x)dx + \int_{0}^{\infty} g(x)\varphi'(x)dx \right) \tag{3.27}$$

$$= - \left(h(x)\varphi(x) \Big|_{-\infty}^{0} - \int_{-\infty}^{0} h'(x)\varphi(x)dx - g(x)\varphi(x) \Big|_{0}^{\infty} - \int_{0}^{\infty} g'(x)\varphi(x)dx \right) \tag{3.28}$$

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where further simplification shows that

$$\langle F_f', \varphi \rangle = (g(0) - h(0))\varphi(0) + \int_{-\infty}^{0} h'(x)\varphi(x)dx + \int_{0}^{\infty} g'(x)\varphi(x)dx \qquad (3.29)$$

$$:= \langle c\delta_0, \varphi \rangle + \langle F_f', \varphi \rangle \qquad (3.30)$$

where $f'(x) = \begin{cases} h'(x) & x < 0 \\ g'(x) & x > 0 \end{cases}$, and $c\delta_0$ just represents that "jump discontinuity".

3.2   Convergence of Functions in Distributions

Definition 10. *A sequence F_n of distributions converges to a distribution F if*

$$\lim_{n \to \infty} \langle F_n, \varphi \rangle = \langle F, \varphi \rangle \quad \text{for all } \varphi \in C_C^{\infty}(\mathbb{R}). \qquad (3.31)$$

We may consider a sequence of locally integrable functions $\{f_n\}$, and we can define its convergence in the sense of distribution:

Definition 11. *A sequence of locally integrable functions $\{f_n\}$ converges to f in the sense of distributions if*

$$\lim_{n \to \infty} \langle F_{f_n}, \varphi \rangle = \langle F_f, \varphi \rangle \quad \text{for all } \varphi \in C_C^{\infty}(\mathbb{R}), \qquad (3.32)$$

which is,

$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n(x)\varphi(x)dx = \int_{\mathbb{R}} f(x)\varphi(x)dx \quad \text{for all } \varphi \in C_C^{\infty}(\mathbb{R}). \qquad (3.33)$$

Proposition 1. *Let $\{F_n\}, F$ be distributions, if $F_n \to F$, then $F_n^{(\alpha)} \to F^{(\alpha)}$ for all higher order derivatives.*

The proof is easy since $\varphi(x) \in C_C^{\infty}(\mathbb{R})$.

Proposition 2. *Let $\{f_n(x)\} \subseteq L_{loc}^1$ be a sequence of locally integrable functions and $f_n \to f$ pointwise where $f \in L_{loc}^1$. Further suppose $\exists g \in L_{loc}^1$ such that $|f_n(x)| \le g(x)$ for all n and $x \in \mathbb{R}$, then $f_n \to f$ in the sense of distributions.*

The proof follows from monotone convergence theorem and dominated convergence theorem. An application to the proposition above is to consider the sequence

$$f_n(x) = \frac{1}{n} e^{-x^2/4n} \qquad (3.34)$$

and it can be shown that $f_n(x) \to 0$ pointwise, so $f_n(x) \to 0$ in the sense of distributions.

Definition 12. *A sequence $\{f_n\} \subseteq L_{loc}^1$ converges in the sense of distributions to δ_0 (the Dirac delta function), if*

$$\int_{\mathbb{R}} f_n(x)\varphi(x)dx \overset{n \to \infty}{\to} \varphi(0) \quad \text{for all } \varphi \in C_C^{\infty}(\mathbb{R}). \qquad (3.35)$$

Proposition 3. *If a sequence $\{f_n\} \subseteq L_{loc}^1$ converges to δ_0 in distribution, then $f_n(x)$ satisfies the following properties:*

*(1) **Nonnegativity:** $f_n(x) \geq 0$ for all $x \in \mathbb{R}, n \in \mathbb{N}$;*

*(2) **Unit mass:** We have $\int_{\mathbb{R}} f_n(x)dx = 1$ for all $n \in \mathbb{N}$.*

*(3) **Concentration at** 0: For any $\varepsilon > 0$, $f_n(x) \to 0$ uniformly on $|x| \geq \varepsilon$.*

Theorem 6. *Let $f(x)$ be any non-negative integrable function on $\mathbb{R}$ and $\int_{\mathbb{R}} f(x)dx = 1$. Define $f_n := nf(nx)$, then $f_n \to \delta_0$ in distribution.*

Proof. We need to show $\langle f_n, \varphi \rangle \to \langle \delta_0, \varphi \rangle := \delta_0$, i.e $\int_{\mathbb{R}} f_n \varphi \to \varphi(0)$ in distribution. distribution. Using $\varepsilon - \delta$ definition, $\forall \varepsilon > 0$, since f_n is non-negative, we have

$$\left| \int_{\mathbb{R}} f_n(x)\varphi(x)dx - \varphi(0) \right| = \left| \int_{\mathbb{R}} f_n(x) \cdot (\varphi(x) - \varphi(0))dx \right| \tag{3.36}$$

$$\leq \int_{\mathbb{R}} f_n(x) \cdot \left| \varphi(x) - \varphi(0) \right| dx \tag{3.37}$$

$$= \int_{\mathbb{R}} f(y) \cdot \left| \varphi\left(\frac{y}{n}\right) - \varphi(0) \right| dy \tag{3.38}$$

By continuity of φ, $\forall \varepsilon > 0$, one can choose δ, such that for $N \in \mathbb{N}$ sufficiently large, $\left| \frac{y}{n} - 0 \right| < \delta$ for all $n \geq N$ would imply $\left| \varphi\left(\frac{y}{n}\right) - \varphi(0) \right| < \varepsilon$, hence

$$\int_{\mathbb{R}} f(y) \cdot \left| \varphi\left(\frac{y}{n}\right) - \varphi(0) \right| dy \leq \varepsilon \int_{\mathbb{R}} f(y)dy \equiv \varepsilon. \tag{3.39}$$

for all $n \geq N$, and hence

$$\left| \int_{\mathbb{R}} f_n(x)\varphi(x) - \varphi(0) \right| < \varepsilon \tag{3.40}$$

for all $n \geq N$ which means $f_n(x) \to \delta_0$ in distribution. ∎

Theorem 7. $\sin(nx) \to 0$ *in distribution.*

Proof. Note that $\varphi \in C_C^\infty(\mathbb{R})$, so $\exists M \in \mathbb{R}^+$ such that $\varphi(x) = 0$ for all $|x| \geq M$, hence

$$\int_{\mathbb{R}} \sin(nx)\varphi(x)dx = \int_{[-M,M]} \sin(nx)\varphi(x)dx. \tag{3.41}$$

Then we have

$$\left| \int_{[-M,M]} \sin(nx)\varphi(x)dx - 0 \right| = \left| \int_{[-M,M]} \left(-\frac{d}{dx} \cdot \frac{\cos(nx)}{n} \right) \varphi(x)dx \right| \tag{3.42}$$

$$= \left| -\frac{\cos(nx)}{n} \cdot \varphi(x) \Big|_{x=-M}^{x=M} + \int_{[-M,M]} \frac{\cos(nx)}{n} \cdot \varphi'(x)dx \right| \tag{3.43}$$

$$\leq \frac{1}{n} \int_{[-M,M]} \left| \cos(nx) \cdot \varphi'(x) \right| dx \tag{3.44}$$

$$\leq \frac{1}{n} \int_{[-M,M]} ||\varphi'(x)||_{L^\infty} dx \tag{3.45}$$

$$= \frac{2M||\varphi'(x)||_{L^\infty}}{n}. \tag{3.46}$$

Simply let $\varepsilon := \frac{2M||\varphi'(x)||_{L^\infty}}{n}$, choose N such that $N > \frac{2M||\varphi'(x)||_{L^\infty}}{\varepsilon}$ so for all $n \geq N$, we have equation (3.36) $< \varepsilon$. ∎

Example 11. *Show that the sequence of function defined by $f_n(x) = \frac{1}{\pi} \cdot \frac{n}{n^2 x^2 + 1}$ converges to δ_0 in distribution.*

Proof. By definition we wish to show that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\left| \int_{\mathbb{R}} f_n(x)\varphi(x)dx - \varphi(0) \right| < \varepsilon \tag{3.47}$$

for all $n \geq \mathbb{N}$. By a similar argument in the proof of **theorem 6** we obtain

$$\left| \int_{\mathbb{R}} f_n(x)\varphi(x)dx - \varphi(0) \right| \leq \int_{\mathbb{R}} f_n(x) \cdot \left| \varphi(x) - \varphi(0) \right| dx. \tag{3.48}$$

Since $\varphi \in C_C^\infty(\mathbb{R})$, so for the ε given, $\exists \delta$ such that $|x| < \delta$ implies $|\varphi(x) - \varphi(0)| < \varepsilon$. We then rewrite the right hand side of (3.42) as

$$\int_{\mathbb{R}} f_n(x) \cdot \left| \varphi(x) - \varphi(0) \right| dx = \underbrace{\int_{[-\delta,\delta]} f_n(x) \cdot \left| \varphi(x) - \varphi(0) \right| dx}_{I} + \underbrace{\int_{\mathbb{R}/[-\delta,\delta]} f_n(x) \cdot \left| \varphi(x) - \varphi(0) \right| dx}_{J} \tag{3.49}$$

where we will investigate I, J separately. For I term, use the fact that $x \in [-\delta, \delta]$, we have

$$I < \varepsilon \int_{[-\delta,\delta]} f_n(x)dx \leq \varepsilon \tag{3.50}$$

since we have $\int_{\mathbb{R}} f_n(x)dx = 1$. For J, we have

$$J \leq \int_{\mathbb{R}/[-\delta,\delta]} f_n(x) \cdot \left(|\varphi(x)| + |\varphi(0)| \right) dx \tag{3.51}$$

$$\leq \int_{\mathbb{R}/[-\delta,\delta]} f_n(x) \cdot 2||\varphi(x)||_{L^\infty} dx \tag{3.52}$$

$$:= \frac{4||\varphi(x)||_{L^\infty}}{\pi} \int_\delta^\infty \frac{n}{n^2 x^2 + 1} dx \tag{3.53}$$

$$= \frac{4||\varphi(x)||_{L^\infty}}{\pi} \cdot \left(\frac{\pi}{2} - \arctan(n\delta) \right) \tag{3.54}$$

by choosing N sufficiently large, we have $\frac{\pi}{2} - \arctan(n\delta) < \varepsilon$ for all $n \geq N$. So all combined, we have

$$\left| \int_{\mathbb{R}} f_n(x)\varphi(x) - \varphi(0) \right| < 2\varepsilon \tag{3.55}$$

for any $\varepsilon > 0$ and sufficiently large $n > N$. Hence $f_n(x) \to \delta_0$ in distribution. ∎

Heat Equations

4.1 Simple Diffusion Equations

In general, let $u(x,t)$ be the function of our interest, heat equation is the PDE that takes the form

$$u_t - k\Delta u = 0. \quad (k > 0) \tag{4.1}$$

We will first study the simple diffusion equation defined by

$$\begin{cases} u_t(x,t) = ku_{xx}(x,t) & \text{in } R \\ u(x,t) = g(x,t) & \text{on } \partial R \end{cases} \tag{4.2}$$

where we restrict the region R to be the rectangle defined by $R := [0,L] \times [0,T]$. It is intuitive to first present where equation (3.2) is coming from. Let us imagine inside a straight tube contains motionless liquid, and a dye is diffusing through the liquid (see the figure below).
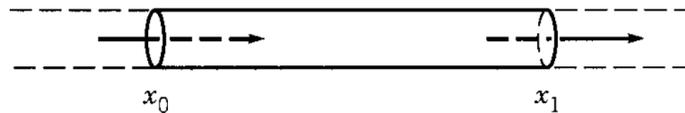

Figure 4: Illustration of our example, where the straight tube is filled with motionless liquid, where a dye is diffusing from x_0 to x_1. *(Source: Strauss PDE)*

By Fick's law the dye will move from the higher concentration region to lower concentration region, and the rate of such a motion is proportional to the concentration gradient. Suppose $u(x,t)$ is the concentration (mass per unit length) of the dye at position x at time t, then in the tube illustrated above, the total mass of the dye is given by

$$M = \int_{x_0}^{x_1} u(x,t)dx \tag{4.3}$$

then applying Fick's law, the rate of change of the concentration (dM) is "flow in from x_0 − flow out from x_1" since we assume the liquid is motionless, hence we have the equation

$$\frac{dM}{dt} = k(u_x(x_0,t) - u_x(x_1,t)) \tag{4.4}$$

where $k > 0$, and combine with (3.3) will yield

$$\int_{x_0}^{x_1} u_t(x,t)dx = k(u_x(x_0,t) - u_x(x_1,t)) \tag{4.5}$$

we differentiate with respect to x_1, and wlog it will yield

$$u_t(x,t) = ku_{xx} \tag{4.6}$$

where k is some constant.

We first study some properties of (3.2), and will intriduce the general solution to the heat equation in the next few sections.

Theorem 8 (Maximum Principle). *If $u(x,t)$ solves (3.2), then*

$$\max_{(u,t)\in R} u(x,t) = \max_{(u,t)\in \partial R} u(x,t) \tag{4.7}$$

The idea of the proof is that, at an interior maximum the first derivative is equal to zero and the second derivative is non-positive. If we knew $u_{xx} \neq 0$ at the maximum then $u_{xx} < 0$ as well as $u_t = 0$, which will contradicts the equation $u_t = ku_{xx}$.

Proof. Let $M := \max_{(u,t)\in\partial R} u(x,t)$. Our goal is to show $u(x,t) \leq M$ for all $(x,t) \in R$. Let $\varepsilon > 0$, let $v(x,t) = u(x,t) + \varepsilon x^2$, then it is clear that $v(x,t) \leq u(x,t) + \varepsilon L^2$ (recall $x \in [0,L]$) for all $(x,t) \in R \cup \partial R$. Further we have

$$v_t - kv_{xx} = u_t - k(u + \varepsilon x^2)_{xx} = u_t - ku_{xx} - 2\varepsilon k = -2\varepsilon k < 0 \tag{4.8}$$

We now suppose $v(x,t)$ attains its maximum at an interior point (x_0,t_0), then we know $v_t = 0, v_{xx} \leq 0$ at (x_0,t_0) and it contradicts the equation above.

We then consider if $v(x,t)$ attains its maximum at $\{t_0 = T, 0 < x < L\}$, then $v_x(x_0,t_0) = 0$ and $v_{xx}(x_0,t_0) \leq 0$. Also since $v(x_0,t_0) \geq v(x_0,t_0 - \delta)$ for $\delta > 0$, then

$$v_t(x_0,t_0) = \lim_{\delta \to 0} \frac{v(x_0,t_0) - v(x_0,t_0 - \delta)}{\delta} \geq 0 \tag{4.9}$$

(Note the maximum is only "one-sided" in the variable t), hence that's a contradiction. ∎

Note that we proved the weak maximum principle, which is the maximum on the boundary is equal to the maximum inside the exterior, but there is a strong version of maximum principle which states the maximum cannot be assumed anywhere inside R but only ∂R. This is much harder to prove.

Theorem 9. *(Uniqueness of solution) Suppose u,v both solves the PDE* $\begin{cases} u_t = ku_{xx} & \text{in } R \\ u = g & \text{on } \partial R \end{cases}$*, then $u = v$ for all $(x,t) \in R$.*

Proof. Let $w = u - v$, where $w_t = kw_{xx}$ in R and $w = g - g$ on ∂R. Then by maximum / minimum principle, we know that

$$\max_{(x,t)\in R} w(x,t) = \max_{(x,t)\in\partial R} w(x,t) = 0 \quad \min_{(x,t)\in R} w(x,t) = \min_{(x,t)\in\partial R} w(x,t) = 0 \tag{4.10}$$

which implies $w \equiv 0$ for all $(x,t) \in R$, hence $u \equiv v$. ∎

Theorem 10 (Dirichlet Problem for the Diffusion Equation). *There is at most one solution of*

$$\begin{cases} u_t - ku_{xx} = f(x,t) & \text{for } 0 < x < l, t > 0 \\ u(x,0) = \varphi(x) \\ u(0,t) = g(t) \qquad u(l,t) = h(t) \end{cases} \tag{4.11}$$

Proof. Let $u_1(x,t), u_2(x,t)$ that bot solves (4.11), lwt $w = u_1 - u_2$, then $w_t - kw_{xx} = 0$, $w(x,0) = w(0,t) = w(l,t) = 0$. Let $T > 0$ then by maximum principle $w(x,t)$ has its maximum for the rectangle on its boundary (bottom ot sides), so we have $w(x,t) \leq 0$. The same argument using minimum principle will yield $w(x,t) \geq 0$, therefore $w(x,t) \equiv 0$ and hence $u_1 \equiv u_2$. ∎

Theorem 11. *(Stability) Suppose u,v satisfies*

$$\begin{cases} u_t = ku_{xx} & \text{in } R \\ u = g_1 & \text{on } \partial R \end{cases} \quad \text{and} \quad \begin{cases} v_t = kv_{xx} & \text{in } R \\ v = g_2 & \text{on } \partial R \end{cases} \tag{4.12}$$

then define $w := u - v$, we have

$$||w||_\infty \leq ||g_1 - g_2||_\infty. \tag{4.13}$$

Proof. By maximum principle, we know $\max_{(x,t) \in R} w(x,t) = \max_{(x,t) \in \partial R}(g_1 - g_2)$, and the result follows. ∎

Proposition 4. *(Energy Method) In $u_t = ku_{xx}$ defined on $R := [0,L] \times [0,T)$, assuming zero boundary condition ($u = 0$ at $x = 0,L$) and no flux boundary condition ($u_x = 0$ at $x = 0,L$), the energy defined by*

$$E(t) = \frac{1}{2} \int_0^L u^2(x,t)dx \tag{4.14}$$

is non-increasing.

Proof. We differentiate the energy with respect to time t and we get

$$\frac{d}{dt}E(t) = \frac{1}{2} \int_0^L \frac{d}{dt}u^2(x,t)dx = \int_0^L u(x,t)u_t(x,t)dx \tag{4.15}$$

by substituting we get

$$\frac{d}{dt}E(t) = \int_0^L u(x,t)ku_{xx}(x,t)dx \tag{4.16}$$

$$= ku(x,t)u_x(x,t)\Big|_{x=0}^{x=L} - k\int_0^L u_x^2(x,t)dx \tag{4.17}$$

$$= -k\int_0^L u_x^2(x,t)dx \tag{4.18}$$

$$\leq 0 \tag{4.19}$$

i.e $E'(t) \leq 0$ so the energy is non-increasing. ∎

4.2   Fundamental Solutions

In this section we will solve the heat equation on the real line $\mathbb{R}$, that is, $x \in \mathbb{R}, t \in [0,\infty)$:

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = \varphi(x) \end{cases} \tag{4.20}$$

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To solve (4.20), we will first introduce some invariant properties:

(i) (4.20) satisfies linearity. if u, v both solves (4.20), then so is $cu + dv$ for any scalar $c, d \in \mathbb{R}$;

(ii) (4.20) is invariant under translation. If $u(x,t)$ is a solution, then for a fixed y, $u(x-y,t)$ is also a solution;

(iii) If $u(x,t)$ solves (4.20), then any of its derivatives (u_x, u_t, u_{xx}, etc.) also solves (4.20);

(iv) Suppose $u(x,t)$ solves (4.20), then so is $u(\lambda x, \lambda^2 t)$ for any $\lambda \in \mathbb{R}$.

(v) An integral solution is again a solution. If $S(x,t)$ is a solution then so is $S(x-y,t)$ and so is

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)dy \tag{4.21}$$

for any function $g(y)$.

Theorem 12 (Fundamental Solution). *The function*

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \tag{4.22}$$

is the fundamental solution to heat equation $u_t = ku_{xx}$ when $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, the solution given by the boundary condition $u_t = ku_{xx}, u(x,0) = \varphi(x)$ is then

$$u(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t)\varphi(y)dy = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot \varphi(y)dy \tag{4.23}$$

Example 12. *Find the solution $u(x,t)$ to the heat equation $u_t = ku_{xx}$ with boundary conditions $\varphi(x) = 1, \forall |x| < l$ and $\varphi(x) = 0$ otherwise.*

We plus in the formula and we have

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot \mathbf{1}_{|y|<l}dy = \frac{1}{\sqrt{4\pi kt}} \int_{-l}^{l} e^{-\frac{(x-y)^2}{4kt}} dy \tag{4.24}$$

let $u = \frac{x-y}{\sqrt{4kt}}$, then $du = -\frac{1}{\sqrt{4kt}}dy$ hence

$$\frac{1}{\sqrt{4\pi kt}} \int_{-l}^{l} e^{-\frac{(x-y)^2}{4kt}} dy = \frac{1}{\sqrt{4\pi kt}} \int_{-\frac{x-l}{\sqrt{4kt}}}^{\frac{x+l}{\sqrt{4kt}}} -(\sqrt{4kt})e^{-u^2} du \tag{4.25}$$

Hence we have

$$u(x,t) = -\frac{1}{\sqrt{\pi}} \int_{-\frac{x-l}{\sqrt{4kt}}}^{\frac{x+l}{\sqrt{4kt}}} e^{-u^2} du = -\frac{1}{2}erf\left(\frac{x-l}{\sqrt{4kt}}\right) + \frac{1}{2}erf\left(\frac{x+l}{\sqrt{4kt}}\right). \tag{4.26}$$

where we define

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-p^2} dp. \tag{4.27}$$

Example 13. *Find the solution $u(x,t)$ to the heat equation $u_t = ku_{xx}$ with boundary condition $\varphi(x) = 1, x > 0$, $\varphi(x) = 3, x < 0$.*

We have that

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{(x-y)^2}{4kt}} dy + \frac{3}{\sqrt{4\pi kt}} \int_{-\infty}^0 e^{-\frac{(x-y)^2}{4kt}} dy \tag{4.28}$$

where we do a similar change of variable and let $u = \frac{x-y}{\sqrt{4kt}}$, so we have

$$du = -\frac{1}{\sqrt{4kt}} dy \tag{4.29}$$

and hence we have

$$u(x,t) = -\frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-u^2} du - \frac{3}{\sqrt{\pi}} \int_{\infty}^{\frac{x}{\sqrt{4kt}}} e^{-u^2} du \tag{4.30}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-u^2} du + \frac{2}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-u^2} du \tag{4.31}$$

$$= 1 + \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} du - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-u^2} du \tag{4.32}$$

$$= 2 - erf\left(\frac{x}{\sqrt{4kt}}\right). \tag{4.33}$$

where we defined

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp. \tag{4.34}$$

We next study some properties of the solution formula

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^\infty e^{-\frac{(x-y)^2}{4kt}} \varphi(y) dy \tag{4.35}$$

Proposition 5. *The fundamental solution $\Phi(x,t)$ satisfies*

$$\int_\infty^\infty \Phi(x,t) dx = 1. \tag{4.36}$$

Proof. The result is obvious using change of variables. ∎

Proposition 6 (Maximum Principle)**.** *For the heat equation with boundary condition $u_t = ku_{xx}$ and $u(x,0) = \varphi(x)$, if there exists a constant B such that $|phi(x)| \le B$ for all $x \in \mathbb{R}$, then $|u(x,t)| \le B$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.*

Proof. We have

$$|u(x,t)| \le \int_{-\infty}^\infty \Phi(x-y,t) \cdot |\varphi(y)| dy \le B \int_{-\infty}^\infty \Phi(x-y,t) dy = B. \tag{4.37}$$

∎

Proposition 7 (Decay)**.** *Suppose $C := \int_{-\infty}^\infty |\varphi(y)| dy < \infty$, then $\lim_{t \to \infty} |u(x,t)| = 0$.*

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Proof. Note that for all $t > 0$ and x, y, we have

$$|u(x,t)| \leq \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} |\varphi(y)| dy \tag{4.38}$$

$$\leq \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} |\varphi(y)| dy \tag{4.39}$$

$$= \frac{C}{\sqrt{4\pi kt}}. \tag{4.40}$$

■

Theorem 13. *Let φ be a bounded, continuous and integrable function on $\mathbb{R}$, define*

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y) dy \tag{4.41}$$

then $u(x,t) \in C^{\infty}$, and $\lim_{t\to 0} u(x,t) = \varphi(x)$. Furthermore, $u(x,t)$ solves the heat equation $u_t = ku_{xx}$.

Proof. We will only proof the convergence here. Note that

$$\left| \int_{-\infty}^{\infty} \Phi(x-y,t)\varphi(y) dy - \varphi(x) \right| \leq \int_{-\infty}^{\infty} \Phi(x-y,t) \cdot |g(y) - g(x)| dy \tag{4.42}$$

Since φ is continuous and bounded, so $\exists B$ such that $|\varphi(x)| \leq B$, also $\forall \varepsilon > 0$, $\exists \varepsilon$ such that $|y - x| < \delta$ implies $|\varphi(y) - \varphi(x)| < \varepsilon$. Hence we have

$$\int_{-\infty}^{\infty} \Phi(x-y,t) \cdot |\varphi(y) - \varphi(x)| dy = \int_{|y-x|<\delta} \Phi(x-y,t) \cdot |\varphi(y) - \varphi(x)| dy$$

$$+ \int_{|y-x|\geq\delta} \Phi(x-y,t) \cdot |\varphi(y) - \varphi(x)| dy \tag{4.43}$$

$$\leq \varepsilon \int_{|y-x|<\delta} \Phi(x-y,t) dy + \int_{|y-x|\geq\delta} \Phi(x-y,t) \cdot |\varphi(y) - \varphi(x)| dy \tag{4.44}$$

$$\leq \varepsilon + \frac{1}{\sqrt{4\pi kt}} \int_{|y-x|\geq\delta} e^{-\frac{(x-y)^2}{4kt}} \cdot |\varphi(y) - \varphi(x)| dy \tag{4.45}$$

$$\leq \varepsilon + \frac{2B}{\sqrt{4\pi kt}} \int_{|y-x|\geq\delta} e^{-\frac{(x-y)^2}{4kt}} dy \tag{4.46}$$

$$= \varepsilon - \frac{2}{\sqrt{\pi}} \int_{|u|\geq\frac{\delta}{\sqrt{4kt}}} e^{-u^2} du. \tag{4.47}$$

$$\tag{4.48}$$

Note that since

$$\lim_{t\to 0} \frac{2}{\sqrt{\pi}} \int_{|u|\geq\frac{\delta}{\sqrt{4kt}}}^{\infty} e^{-u^2} du = 0 \tag{4.49}$$

we then have

$$\left| \int_{-\infty}^{\infty} \Phi(x-y,t)\varphi(y) dy - \varphi(x) \right| < \varepsilon. \tag{4.50}$$

■

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Laplace Equations

5.1 Properties of Harmonic Functions

Definition 13. *Let $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, u is said to be harmonic if $\Delta u = 0$, where*

$$\Delta u = \sum_{n=1}^{n} u_{x_i x_i} \tag{5.1}$$

In this section, we study the PDE

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial \Omega \end{cases} \tag{5.2}$$

and we will mainly focus on $u : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$.

Proposition 8 (Mean Value Property). *Let u be a C^2 harmonic function on $\Omega \subseteq \mathbb{R}^3$, let $x_0 \in \Omega$, $B(x_0, r) \subseteq \Omega$, then*

$$u(x_0) = \frac{1}{4\pi r^2} \iint_{\partial B(x_0, r)} u(x) dS(x) \tag{5.3}$$

Proof. We define a function

$$\varphi(r) = \frac{1}{4\pi r^2} \iint_{\partial B(x_0, r)} u(x) dS(x) \tag{5.4}$$

since $u \in C^2$, by averaging lemma

$$\lim_{r \to 0} \varphi(r) = \lim_{r \to 0} \frac{1}{4\pi r^2} \iint_{\partial B(x_0, r)} u(x) dS(x) = u(x_0) \tag{5.5}$$

and we now show $\varphi(r)$ is constant in Ω by taking the derivative. Let $y = \frac{x - x_0}{r}$, then $dS(y) = \frac{1}{r^2} dS(x)$ hence we have

$$\varphi(r) = \lim_{r \to 0} \frac{1}{4\pi r^2} \iint_{\partial B(x_0, r)} u(x) dS(x) = \frac{1}{4\pi} \iint_{\partial B(0, 1)} u(x_0 + ry) dS(y) \tag{5.6}$$

now we have

$$\varphi'(r) = \frac{1}{4\pi} \iint_{\partial B(0, 1)} \frac{d}{dr} u(x_0 + ry) dS(y) = \frac{1}{4\pi} \iint_{\partial B(0, 1)} \nabla u(x_0 + ry) y \, dS(y) \tag{5.7}$$

$$(\text{revert back to original coordinates}) = \frac{1}{4\pi r^2} \iint_{\partial B(x_0, r)} \nabla u(x) \cdot \frac{x - x_0}{r} dS(x) \tag{5.8}$$

and note that $\frac{x - x_0}{r}$ is the outer unit normal to $\partial B(x_0, r)$ hence by divergence theorem, we have

$$\frac{1}{4\pi r^2} \iint_{\partial B(x_0, r)} \nabla u(x) \cdot \frac{x - x_0}{r} dS(x) = \frac{1}{4\pi r^2} \iiint_{B(x_0, r)} \Delta u(x) dx \equiv 0 \tag{5.9}$$

since u is harmonic. Hence $\varphi(r)$ is a constant and is equal to $\varphi(0) = u(x_0)$. ∎

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We may also averaging around a ball $B(x_0, r) \subseteq \Omega$, and we have

$$u(x_0) = \frac{1}{\frac{4}{3}\pi r^3} \iiint_{B(x_0,r)} u(x)dx. \tag{5.10}$$

Corollary 1 (Mean Value Property in $\mathbb{R}^2$). *Let u be a C^1 harmonic function on $\Omega \subseteq \mathbb{R}^2$, let $x_0 \in \Omega$, $B(x_0, r) \subseteq \Omega$, then*

$$u(x_0) = \frac{1}{2\pi r} \int_{\partial B(x_0,r)} u(x)dx \tag{5.11}$$

Proof. Similarly define

$$\varphi(r) = \frac{1}{2\pi r} \int_{\partial B(x_0,r)} u(x)dx \quad \varphi(0) = u(x_0) \tag{5.12}$$

and by change of variable

$$y = \frac{x - x_0}{r}, \quad \varphi(r) = \frac{1}{2\pi} \int_{\partial B(0,1)} u(x_0 + ry)\, dy \tag{5.13}$$

now by taking derivative

$$\varphi'(r) = \frac{1}{2\pi} \frac{d}{dr} u(x_0 + ry)dy \tag{5.14}$$

$$= \frac{1}{2\pi} \int_{\partial B(0,1)} y\nabla u(x_0 + ry)dy \tag{5.15}$$

$$= \frac{1}{2\pi r} \int_{\partial B(x_0,r)} \frac{x - x_0}{r} \nabla u(x)dx \tag{5.16}$$

$$= \frac{1}{2\pi r} \iint_{B(x_0,r)} \Delta u(x)dx = 0. \tag{5.17}$$

∎

Proposition 9 (Maximum Principle). *Let u be a C^2 harmonic function on a bounded domain $\Omega \subseteq \mathbb{R}^3$ and $u \in C(\bar{\Omega})$. if u attains its maximum at a point in Ω, then u must be identically constant inside $\bar{\Omega}$*

Corollary 2 (Uniqueness). *Let $\Omega \subseteq \mathbb{R}^3$ be bounded, then there exists at most one $u \in C^2(\Omega) \cap C(\bar{\Omega})$ that solves te Dirichlet problem*

$$\begin{cases} \Delta u = 0 & in\ \Omega \\ u = g & on\ \partial\Omega \end{cases} \tag{5.18}$$

Corollary 3 (Stability). *Let $\Omega \subseteq \mathbb{R}^3$ bounded, $g_1, g_2 \in C(\partial\Omega)$, $u_1, u_2 \in C^2(\Omega) \cup C(\bar{\Omega})$ and they solves*

$$\begin{cases} \Delta u_1 = 0 & in\ \Omega \\ u_1 = g_1 & on\ \partial\Omega \end{cases} \quad \begin{cases} \Delta u_2 = 0 & in\ \Omega \\ u_2 = g_2 & on\ \partial\Omega \end{cases} \tag{5.19}$$

then

$$\max_{x \in \Omega} |u_1(x) - u_2(x)| \le \max_{x \in \partial\Omega} |g_1(x) - g_2(x)|. \tag{5.20}$$

Proposition 10 (Dirichlet's Principle). *Let $\Omega \subseteq \mathbb{R}^3$, define a class of functions*

$$A_h = \left\{ w \in C^2(\Omega) \cap C(\bar{\Omega}) : w = h \text{ on } \partial\Omega \right\} \tag{5.21}$$

for some function h, then define

$$E(w) = \frac{1}{2} \iiint_\Omega |\nabla w|^2 dx \tag{5.22}$$

then let $u \in A_h$, $E(u) \leq E(w)$ for all $w \in A_h$ if and only if u solves the PDE

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = h & \text{on } \partial\Omega \end{cases} \tag{5.23}$$

Proof. ($\Longrightarrow$): First suppose u is a minimizer over all $w \in A_h$. Let $v \in C^2(\Omega) \cap C(\bar{\Omega})$ such that $v = 0$ on $\partial\Omega$, then $u + \varepsilon v \in A_h$ for all $\varepsilon > 0$. Define

$$f(\varepsilon) = E(u + \varepsilon v) = \frac{1}{2} \iiint_\Omega |\nabla(u + \varepsilon v)|^2 dx. \tag{5.24}$$

Since f is differentiable, we would have

$$f'(\varepsilon) = \frac{1}{2} \iiint_\Omega \frac{d}{d\varepsilon} (|\nabla u|^2 + 2\varepsilon \nabla u \cdot \nabla v + \varepsilon^2 |\nabla v|^2) dx \tag{5.25}$$

$$= \frac{1}{2} \iiint_\Omega (2\nabla u \cdot \nabla v + 2\varepsilon |\nabla v|^2) dx \tag{5.26}$$

$$\tag{5.27}$$

because u is a minimizer so $f'(0) = 0$ which implies

$$0 = f'(0) = \iiint_\Omega \nabla u \cdot \nabla v \, dx = - \iiint_\Omega (\Delta u) v \, dx \tag{5.28}$$

and hence $\Delta u = 0$ for all $x \in \Omega$.

($\Longleftarrow$): Suppose $u \in A_h$ solves the Dirichlet's problem, let $w \in A_h$, define $v = u - w$, then

$$E(w) = E(u - v) = \frac{1}{2} \iiint_\Omega |\nabla(u - v)|^2 dx = E(u) - \iiint_\Omega \nabla u \cdot \nabla v \, dx + E(v) \tag{5.29}$$

use the fact that $v = 0$ on $\partial\Omega$, then

$$0 = \iint_{\partial\Omega} v \cdot \frac{\partial u}{\partial n} dS = \iiint_\Omega \nabla u \cdot \nabla v \, dx + \iiint_\Omega v \Delta u \, dx := \iiint_\Omega \nabla u \cdot \nabla v \, dx \tag{5.30}$$

which implies $E(w) = E(u) + E(v)$, and $E(v) \geq 0$. $\blacksquare$

Corollary 4 (Dirichlet's principle for Poisson's equation). *Define*

$$E(w) := \iiint_\Omega \frac{1}{2} |\nabla w|^2 + wf \, dx \tag{5.31}$$

then the last theorem still holds for the Poisson's equation

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = h & \text{on } \partial\Omega. \end{cases} \tag{5.32}$$

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Definition 14. *The fundamental solution for Laplace equation $\Delta u = 0$ in $\mathbb{R}^n$ is given by the form*

$$\Phi(x) = \begin{cases} \frac{1}{2\pi}\log|x| & n=2 \\ -\frac{1}{4\pi|x|} & n=3 \\ -\frac{1}{n(n-2)\omega_n|x|^{n-2}} & n>3 \end{cases} \tag{5.33}$$

5.2 Green's Function

Some remarks on calculus:

Theorem 14 (Divergence Theorem). *Let $\mathbf{F}$ be a smooth vector field on a bounded domain Ω with outer normal $\mathbf{n}$, then*

$$\iiint_\Omega \nabla \cdot \mathbf{F} dV = \iint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} dS \tag{5.34}$$

Pointwise Divergence Theorem: Given a smooth function $f(x)$ on a bounded domain $\Omega \subseteq \mathbb{R}^3$, then

$$\iiint_\Omega f_{x_i}(x) dx = \iint_{\partial\Omega} f_{n_i} dS \tag{5.35}$$

where n_i is the i-th component of the outer unit normal $\mathbf{n}$.

Theorem 15 (Integration by Parts).

$$\iint_\Omega \mathbf{u} \cdot \nabla v dx = -\iiint_\Omega (\nabla \cdot \mathbf{u}) v dx + \iint_{\partial\Omega} (v\mathbf{u}) \cdot \mathbf{n} dS \tag{5.36}$$

Theorem 16 (Green's Identities).

$$\iiint_\Omega v\Delta u = \iint_{\partial\Omega} v\frac{\partial u}{\partial \mathbf{n}} - \iiint_\Omega \nabla u \cdot \nabla v \tag{5.37}$$

$$\iiint_\Omega (v\Delta u - u\Delta v) = \iint_{\partial\Omega} v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}} \tag{5.38}$$

We mainly examine the case for $n=3$. Recall in $\mathbb{R}^3$ the fundamental solution is given by $\Phi(x) = -\frac{1}{4\pi|x|}$.

Theorem 17. *Let $n=3$, then $\Delta\left(\frac{1}{|x|}\right) = -4\pi\delta_0$ in distribution. That is,*

$$-4\pi\varphi(0) = \iiint_{\mathbb{R}^3} \frac{1}{|x|}\Delta\varphi(x) dx \tag{5.39}$$

for all $\varphi \in C_C^\infty(\mathbb{R}^3)$.

Proof. Let $B(0,\varepsilon)$ be a small ε-ball centered at the origin, then

$$-4\pi\varphi(0) = \iiint_{B(0,\varepsilon)} \frac{1}{|x|}\Delta\varphi(x) dx + \iiint_{\mathbb{R}^3\setminus B(0,\varepsilon)} \frac{1}{|x|}\Delta\varphi(x) dx \tag{5.40}$$

Since $\varphi \in C_C^\infty$, so we may restrict the region to a compactly support set Ω, such that $\Delta \varphi(x) \equiv 0$ for all $x \notin \Omega$ (including $\partial \Omega$), i.e

$$-4\pi\varphi(0) = \underbrace{\iiint_{B(0,\varepsilon)} \frac{1}{|x|}\Delta\varphi(x)dx}_{\text{equation 1}} + \underbrace{\iiint_{\Omega\backslash B(0,\varepsilon)} \frac{1}{|x|}\Delta\varphi(x)dx}_{\text{equation 2}} \tag{5.41}$$

Now for equation 1, assume $|\Delta\varphi(x)| \leq K_1$ for all $x \in B(0,\varepsilon)$, we have

$$\iiint_{B(0,\varepsilon)} \frac{1}{|x|}\Delta\varphi(x)dx \leq \iiint_{B(0,\varepsilon)} \frac{1}{|x|} \cdot |\Delta\varphi(x)|dx \tag{5.42}$$

$$\leq K_1 \iiint_{B(0,\varepsilon)} \frac{1}{|x|}dx \tag{5.43}$$

$$= K_1 \int_0^\varepsilon \frac{1}{r} \cdot 4\pi r^2 dr \tag{5.44}$$

$$= 2K_1\pi\varepsilon^2. \tag{5.45}$$

hence

$$\lim_{\varepsilon \to 0} \iiint_{B(0,\varepsilon)} \frac{1}{\Delta\varphi(x)}dx = 0. \tag{5.46}$$

Now for equation 2, we apply Green's identity on the two boundaries $\partial B(0,\varepsilon)$ and $\partial \Omega$ and we have

$$\iiint_{\Omega\backslash B(0,\varepsilon)} \left(\frac{1}{|x|}\Delta\varphi(x) - \varphi(x)\Delta\frac{1}{|x|} \right) dx = \underbrace{\iint_{\partial\Omega} \left(\frac{1}{|x|}\frac{\partial\varphi}{\partial\mathbf{n}} - \varphi\frac{\partial\frac{1}{|x|}}{\partial\mathbf{n}} \right) dS}_{\text{equation 3}} \tag{5.47}$$

$$+ \underbrace{\iint_{\partial B(0,\varepsilon)} \left(\frac{1}{|x|}\frac{\partial\varphi}{\partial\mathbf{n}} - \varphi\frac{\partial\frac{1}{|x|}}{\partial\mathbf{n}} \right) dS}_{\text{equation 4}} \tag{5.48}$$

by our choice of Ω, $\varphi(x) \equiv 0$ for all $x \in \partial\Omega$ as well as all its derivatives, so equation 3 is identically zero, hence

$$\iiint_{\Omega\backslash B(0,\varepsilon)} \left(\frac{1}{|x|}\Delta\varphi(x) - \varphi(x)\Delta\frac{1}{|x|} \right) dx = \iint_{\partial B(0,\varepsilon)} \left(\frac{1}{|x|}\frac{\partial\varphi}{\partial\mathbf{n}} - \varphi\frac{\partial\frac{1}{|x|}}{\partial\mathbf{n}} \right) dS. \tag{5.49}$$

Note that

$$\left| \frac{\partial\varphi}{\partial\mathbf{n}} \right| = |\nabla\Phi \cdot \mathbf{n}| \leq |\nabla\varphi|, \tag{5.50}$$

assume $|\nabla\varphi| \leq K_2$ for all $x \in \partial B(0,\varepsilon)$, now we have

$$\iint_{\partial B(0,\varepsilon)} \frac{1}{|x|}\frac{\partial\varphi}{\partial\mathbf{n}} \leq \iint_{\partial B(0,\varepsilon)} \frac{1}{|x|}\left|\frac{\partial\varphi}{\partial\mathbf{n}}\right|dS \tag{5.51}$$

$$\leq K_2 \iint_{\partial B(0,\varepsilon)} \frac{1}{|x|}dS \tag{5.52}$$

$$= K_2 \iint_{\partial B(0,\varepsilon)} \frac{1}{\varepsilon}dS \tag{5.53}$$

$$= 4K_2\pi\varepsilon \tag{5.54}$$

hence
$$\lim_{\varepsilon \to 0} \iint_{\partial B(0,\varepsilon)} \frac{1}{|x|} \frac{\partial \varphi}{\partial \mathbf{n}} dS = 0. \tag{5.55}$$

Finally we study

$$\iint_{\partial B(0,\varepsilon)} \varphi \frac{\partial}{\partial \mathbf{n}} \frac{1}{|x|} dS = \iint_{\partial B(0,\varepsilon)} -\varphi \frac{\partial}{\partial r} \frac{1}{r} dS \tag{5.56}$$

$$= \iint_{\partial B(0,\varepsilon)} -\frac{\varphi(x)}{\varepsilon^2} dS \tag{5.57}$$

$$= -4\pi \left(\frac{1}{4\pi\varepsilon^2} \iint_{\partial B(0,\varepsilon)} \varphi(x) dS \right). \tag{5.58}$$

Eventually we put all terms together:

$$\iiint_{\mathbb{R}^3} \frac{1}{|x|} \Delta \varphi(x) dx = \lim_{\varepsilon \to 0} -4\pi \left(\frac{1}{4\pi\varepsilon^2} \iint_{\partial B(0,\varepsilon)} \varphi(x) dS \right) := -4\pi \varphi(0). \tag{5.59}$$

$\blacksquare$

Boundary Problems

6.1   Separation of Variables and the Dirichlet Condition

We first consider the homogeneous Dirichlet conditions for the wave equation:

$$u_{tt} = c^2 u_{xx} \quad (0 < x < l), \quad u(0,t) = 0 = u(l,t) \tag{6.1}$$

with initial conditions $u(x,0) = \varphi(x), u_t(x,0) = \psi(x)$. The method we shall use consists of building up the general solution as a linear combination of special ones that are easy to find. We say a *separated solution* is a solution of the wave equation defined above taking the form

$$u(x,t) = X(x)T(t) \tag{6.2}$$

by assuming a solution of the form (6.2), we have

$$X(x)T''(t) = c^2 X''(x)T(t) \implies -\frac{T''}{c^2 T} = -\frac{X''}{X} = \lambda \tag{6.3}$$

where λ is a constant since $\frac{\partial \lambda}{\partial x} = \frac{\partial \lambda}{\partial t} = 0$, we will later show that $\lambda > 0$, so define $\lambda = \beta^2, \beta > 0$, so we have a pair of separable ODEs:

$$X'' + \beta^2 X = 0 \quad T'' + c^2 \beta^2 T = 0 \tag{6.4}$$

which leads to the solution

$$X(x) = C\cos\beta x + D\sin\beta x \quad T(t) = A\cos\beta ct + B\sin\beta ct \tag{6.5}$$

we now want to impose boundary condition that $u(0,t) = u(l,t) = 0$, i.e $X(0) = X(l) = 0$, so $X(0) = C, X(l) = D\sin\beta l$, and $\beta l = n\pi$, so for each n we have a separate solution for X, denote by $X_n(x)$, and so $u_n(x,t) = X_n(x)T(t)$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, X_n(x) = \sin\frac{n\pi x}{l} \tag{6.6}$$

and

$$u_n(x,t) = \left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \tag{6.7}$$

so we can write the general solution as a sum of u_n:

$$u(x,t) = \sum_n \left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}. \tag{6.8}$$

This formula will solve the wave equation with boundary condition

$$u_{tt} = c^2 u_{xx}, u(0,t) = u(l,t) = 0, u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \tag{6.9}$$

if

$$\varphi(x) = \sum_n A_n\sin\frac{n\pi x}{l} \quad \psi(x) = \sum_n \frac{n\pi c}{l}B_n\sin\frac{n\pi x}{l} \tag{6.10}$$

Later, we will see Fourier series, and then (6.10) will become more clear.

We may also consider the diffusion equation

$$\textbf{DE:}u_t = ku_{xx}(0 < x < l, 0 < t < \infty), \quad \textbf{BC:}u(0,t) = u(l,t) = 0, \quad \textbf{IC:}u(x,0) = \varphi(x) \tag{6.11}$$

we may separate variables as before, let $u(x,t) = X(x)T(t)$ and get the following

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda \tag{6.12}$$

So we have $T' = -\lambda kT$ and the solution is given by $T(t) = Ae^{-\lambda kt}$, also $-X'' = \lambda X$ is the exact same equation we get as in the wave equation, so in general we have

$$u(x,t) = \sum_n A_n e^{(n\pi/l)^2 kt} \sin \frac{n\pi x}{l} \tag{6.13}$$

and this solves (6.11) if

$$\varphi(x) = \sum_n A_n \sin \frac{n\pi x}{l}. \tag{6.14}$$

The numbers $\lambda_n = (n\pi/l)^2$ are called the *eigenvalues*, the functions $X_n(x) = \sin(n\pi x/l)$ are called the *eogenfunctions*. The reason is that they satisfy

$$-\frac{d^2}{dx^2}X = \lambda X, \quad X(0) = X(l) = 0 \tag{6.15}$$

Fourier Series

7.1 The Coefficients for Sine and Cosine Series

Definition 15. *On the interval $(0,l)$, we define the Fourier sine series as*

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \tag{7.1}$$

Later we will show that for any function defined $\varphi(x)$ defined on $(0,l)$ then the Fourier since series converges to $\varphi(x)$. Our task is try to find the coefficients A_n first.

Lemma 1. *If $m \neq n$, then*

$$\int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = 0 \tag{7.2}$$

Proof. We use the trig-identity that

$$\sin(x+y) = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y). \tag{7.3}$$

∎

Now fix m, we have

$$\int_0^l \varphi(x) \sin \frac{m\pi x}{l} dx = \int_0^l \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx \tag{7.4}$$

by interchaning sum and integral (we may do that), and use the fact that when $m \neq n$ the integral is indeed zero, we have

$$\int_0^l \varphi(x) \sin \frac{m\pi x}{l} dx = A_m \int_0^l \sin^2 \frac{m\pi x}{l} dx \tag{7.5}$$

and by direct computation we have

$$\int_0^l \sin^2 \frac{m\pi x}{l} dx = \int_0^l \frac{1}{2} - \frac{1}{2}\cos \frac{2m\pi x}{l} dx \tag{7.6}$$

$$= \left(\frac{1}{2}x - \frac{l}{4m\pi} \sin \frac{2m\pi x}{l} \right) \Bigg|_{x=0}^{l} \tag{7.7}$$

$$= \frac{1}{2}l. \tag{7.8}$$

Hence we have the coefficients of Fourier sine series given by

$$\boxed{A_m = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{m\pi x}{l} dx.} \tag{7.9}$$

Recall the wave equation with Dirichlet conditions:

$$u_{tt} = c^2 u_{xx}, (0 < x < l), \quad u(0,t) = u(l,t) = 0, \quad u(x,0) = \varphi(x), y_t(x,0) = \psi(x), \tag{7.10}$$

if

$$\varphi(x) = \sum_n A_n \sin \frac{n\pi x}{l} \qquad \psi(x) = \sum_n \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l} \tag{7.11}$$

then the solution takes the form

$$u(x,t) = \sum_n \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}. \tag{7.12}$$

where now we can use the formula derived to find the coefficients of A_n, B_n:

$$A_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{m\pi x}{l} dx \qquad \frac{m\pi c}{l} B_m = \frac{2}{l} \int_0^l \psi(x) \sin \frac{m\pi x}{l} dx. \tag{7.13}$$

Definition 16. *We now define the Fourier cosine series on the interval $(0,l)$ by*

$$\varphi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} \tag{7.14}$$

Similarly, if $m \neq n$ we also have the property that

$$\int_0^l \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} dx = 0 \tag{7.15}$$

and we can see that

$$\int_0^l \varphi(x) \cos \frac{m\pi x}{l} dx = A_m \int_0^l \cos^2 \frac{m\pi x}{l} dx = \begin{cases} \frac{1}{2} l A_m & m \neq 0 \\ \frac{1}{2} l A_0 & m = 0 \end{cases} \tag{7.16}$$

Hence the coefficients for the cosine series are given by

$$\boxed{A_m = \frac{2}{l} \int_0^l \varphi(x) \cos \frac{m\pi x}{l} dx}. \tag{7.17}$$

Definition 17. *Now, the full Fourier series of $\varphi(x)$ on the interval $-l < x < l$ is defined as*

$$\varphi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l} \right). \tag{7.18}$$

where

$$\boxed{A_n = \frac{1}{l} \int_{-l}^l \varphi(x) \cos \frac{n\pi x}{l} dx} \tag{7.19}$$

$$\boxed{B_n = \frac{1}{l} \int_{-l}^l \varphi(x) \sin \frac{n\pi x}{l} dx} \tag{7.20}$$

Example 14. *Let $\varphi(x) = 1$ in the interval $(0,l)$, we now find its Fourier series:*

First the Fourier sine series of $\varphi(x) = 1$ is given by

$$1 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \tag{7.21}$$

where using the formula for the coefficients, we have

$$A_n = \frac{2}{l} \int_0^l 1 \cdot \sin \frac{n\pi x}{l} dx = -\frac{2}{n\pi} \cos \frac{n\pi x}{l} \Big|_{x=0}^{x=l} = \begin{cases} 0 & n \text{ is even} \\ \frac{4}{n\pi} & n \text{ is odd} \end{cases}. \tag{7.22}$$

Thus we ca express $\varphi(x) = 1$ on $(0,l)$ as

$$1 = \frac{4}{\pi} \left(\sin \frac{\pi x}{l} + \frac{1}{3} \sin \frac{3\pi x}{l} + \frac{1}{5} \sin \frac{5\pi x}{l} + \cdots \right). \tag{7.23}$$

we may also compute the coefficients for the Fourier cosine series:

$$A_n = \frac{2}{l} \int_0^l \cos \frac{n\pi x}{l} dx = \frac{2}{m\pi} \sin \frac{m\pi x}{l} \Big|_{x=0}^{x=l} \equiv 0, m \neq 0 \tag{7.24}$$

hence the Fourier cosine series is trivial:

$$1 = 1 \tag{7.25}$$

Example 15. *Let $\varphi(x) = x$ in the interval $(0,l)$. We now find its Fourier series:*

First the Fourier sine series is given by

$$x = \sum_{n=1}^{\infty} A_n \sin \frac{\pi n x}{l} \tag{7.26}$$

where the coefficients are given by

$$A_n = \frac{2}{l} \int_0^l x \sin \frac{\pi n x}{l} dx \tag{7.27}$$

note that using integration by parts we have

$$\int_0^l x \sin \frac{\pi n x}{l} dx = -\frac{l}{\pi n} x \cos \frac{\pi n x}{l} \Big|_0^l + \int_0^l \frac{l}{\pi n} \cos \frac{\pi n x}{l} dx \tag{7.28}$$

$$= (-1)^{n+1} \frac{l^2}{\pi n} + \frac{l^2}{\pi^2 n^2} \sin \frac{\pi n x}{l} \Big|_{x=0}^{x=l} \tag{7.29}$$

$$= (-1)^{n+1} \frac{l^2}{\pi n}. \tag{7.30}$$

Hence we have

$$A_n = (-1)^{n+1} \frac{2l}{\pi n} \tag{7.31}$$

and we may express $\varphi(x) = x$ as its Fourier sine series:

$$x = \frac{2l}{\pi} \left(\sin \frac{\pi x}{l} - \frac{1}{2} \sin \frac{2\pi x}{l} + \frac{1}{3} \sin \frac{3\pi x}{l} - \cdots \right). \tag{7.32}$$

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we now express $\varphi(x) = x$ as a Fourier cosine series:

$$x = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{\pi n x}{l} \tag{7.33}$$

and the formula gives us

$$A_0 = \frac{2}{l} \int_0^l x\,dx = l \tag{7.34}$$

and when $n \geq 1$ we have

$$A_n = \frac{2}{l} \int_0^l x \cos \frac{\pi n x}{l}\,dx \tag{7.35}$$

$$= \frac{2}{l} \left[\frac{l}{\pi n} x \sin \frac{\pi n x}{l} \Big|_{x=0}^{x=l} - \int_0^l \frac{l}{\pi n} \sin \frac{\pi n x}{l}\,dx \right] \tag{7.36}$$

$$= \frac{2}{l} \left[\frac{l^2}{\pi^2 n^2} \cos \frac{\pi n x}{l} \Big|_{x=0}^{x=l} \right] \tag{7.37}$$

$$= \begin{cases} -\frac{4l}{\pi^2 n^2} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases} \tag{7.38}$$

so we have the Fourier cosine series of $\varphi(x) = x$ given by

$$x = \frac{l}{2} - \frac{4l}{\pi^2} \left(\frac{1}{4} \cos \frac{\pi x}{l} + \frac{1}{9} \cos \frac{3\pi x}{l} + \frac{1}{25} \cos \frac{5\pi x}{l} + \cdots \right). \tag{7.39}$$

Finally we find the full Fourier series of $\varphi(x) = x$ on the interval $[-l, l]$. Recall that

$$\varphi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l} \right). \tag{7.40}$$

where

$$A_0 = \frac{1}{l} \int_{-l}^{l} x\,dx = 0 \tag{7.41}$$

$$A_n = \frac{1}{l} \int_{-l}^{l} x \cos \frac{n\pi x}{l}\,dx = 0, \quad \text{since it is odd function.} \tag{7.42}$$

$$B_n = \frac{1}{l} \int_{-l}^{l} x \sin \frac{n\pi x}{l}\,dx \tag{7.43}$$

$$= \frac{1}{l} \left(-x \frac{l}{m\pi} \cos \frac{m\pi x}{l} \Big|_{x=-l}^{l} + \int_l^l \frac{l}{m\pi} \cos \frac{m\pi x}{l}\,dx \right) \tag{7.44}$$

$$= (-1)^{n+1} \frac{2l}{n\pi}. \tag{7.45}$$

where we see that it is exactly the same as the Fourier sine series on $(0, l)$. This is because sine and its Fourier sine series are odd.

Example 16. *We will now use Fourier series to solve the wave equation ($0 < x < l$) with boundary conditions :*

$$u_{tt} = c^2 u_{xx}, \quad u(0,t) = u(l,t) = 0, \quad u(x,0) = x, u_t(x,0) = 0. \tag{7.46}$$

We already know in the previous section (boundary problems), that the solution $u(x,t)$ takes the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \tag{7.47}$$

hence we have

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} \left(-A_n \sin \frac{n\pi ct}{l} + B_n \cos \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}. \tag{7.48}$$

we equate the above with boundary conditions, let $t = 0$ we have

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l} = 0 \tag{7.49}$$

so $B_n \equiv 0$. Also

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \tag{7.50}$$

and we already know the coefficients of A_n given from the previous examples. So

$$u(x,t) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}. \tag{7.51}$$

We now derive the complex form of Fourier series. Recall that by *DeMoivre* formula:

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{7.52}$$

so one may use $e^{+in\pi x/l}, e^{-in\pi x/l}$ as an alternative pair of sine and cosine.

Definition 18. *The full Fourier series of $\varphi(x)$ defined on $(-l,l)$ is given by*

$$\varphi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \tag{7.53}$$

and the coefficients can be computed by

$$c_n = \frac{1}{2l} \int_{-l}^{l} \varphi(x) e^{-in\pi x/l} dx \tag{7.54}$$

7.2 Orthogonality and General Fourier Series

Definition 19. *Let $f, g \in C([a,b])$ be two real-valued functions, their inner product is defined as*

$$\langle f, g \rangle \equiv \int_a^b f(x) g(x) dx \tag{7.55}$$

and we say $f(x), g(x)$ are orthogonal if $\langle f, g \rangle = 0$.

Definition 20. *With the inner product defined in the previous definition, we define the L^2 norm of a function $f(x)$ as*

$$||f(x)||^2 := \langle f(x), f(x) \rangle = \int_a^b f^2(x)dx. \tag{7.56}$$

We wish to study the linear operator

$$A = -\frac{d^2}{dx^2} \tag{7.57}$$

with some boundary conditions. Some common boundary conditions are as follows:

- **Dirichlet Boundary Condition:** A function $X(x)$ on $[a,b]$ such that $X(a) = X(b) = 0$;

- **Neumann Boundary Condition:** A function $X(x)$ on $[a,b]$ such that $X'(a) = X'(b) = 0$;

- **Periodic Boundary Condition:** A function $X(x)$ on $[a,b]$ such that $X(a) = X(b); X'(a) = X'(b)$.

To generalize the idea of eigrnvalues and eigenvectors, we wish to find a constant $\lambda \in \mathbb{R}$ and a non-zero function X satisfying the boundary conditions and such that

$$AX = \lambda X. \tag{7.58}$$

We say such a function X is an *eigenfunction* and λ is the associated *eigenvalue*. Also, **the eigenfunctions corresponding to different eigenvalues are orthogonal.** To see this, let X_1, X_2 be two different eigenfunctions satisfying the same boundary condition, then

$$-X_1'' = \lambda_1 X_1 \quad -X_2'' = \lambda_2 X_2 \tag{7.59}$$

and we assume $\lambda_1 \neq \lambda_2$, then we need to show

$$\int_a^b X_1 X_2 dx = 0 \tag{7.60}$$

which is equivalent as showing

$$(\lambda_1 - \lambda_2) \int_a^b X_1 X_2 dx = \int_a^b (-X_1'' X_2 + X_2'' X_1) dx = 0 \tag{7.61}$$

the right hand side of the equation above can be simplified using *Green's second identity*, and equipped with boundary conditions, it will simplify to zero in all three cases.

In general, we consider the following boundary conditions:

$$\alpha_1 X(a) + \beta_2 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0 \tag{7.62}$$

$$\alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0 \tag{7.63}$$

Definition 21. *Such a set of boundary conditions defined as above is called symmetric if*

$$f'(x)g(x) - f(x)g'(x) \Big|_a^b = 0 \tag{7.64}$$

where $f(x), g(x)$ satisfy the boundary condition defined above.

Theorem 18. *With symmetric boundary condtions, any two eigenfunctions that correspond to different eigenvalues are orthogonal.*

Now let λ_n be the eigenvalue of the eigenfunction $X_n(x)$, we write

$$\varphi(x) = \sum_n A_n X_n(x) \tag{7.65}$$

then

$$\langle \varphi(x), X_m(x) \rangle = \sum_n A_n \langle X_n, X_m \rangle = A_m \langle X_m, X_m \rangle \tag{7.66}$$

by orthogonality.

Definition 22. *Suppose we have symmetric boundary conditions, and for a function $\varphi(x)$ satisfy the form $\varphi(x) = \sum_n A_n X_n(x)$, then*

$$A_n = \frac{\langle \varphi(x), X_n(x) \rangle}{||X_n(x)||^2} = \frac{\int_a^b \varphi(x) X_n(x) dx}{\int_a^b X_n^2 dx} \tag{7.67}$$

is called the general Fourier series of the function $\varphi(x)$.

Theorem 19. *All eigenvalues of the eigenvalue problem $-X'' = \lambda X = 0$ over functions $X(x), x \in [a,b]$ satisfying symmetric boundary conditions are real.*

Proof. Assume $\lambda \in \mathbb{C}$ is an eigenvalue of $X(x)$, then by complex conjugate, $\bar{\lambda}$ is also an eigenvalue of $\bar{X}$. Hence

$$\langle \bar{X}, \lambda X \rangle = \langle \bar{\lambda} \bar{X}, X \rangle \tag{7.68}$$

which means we have

$$\int_a^b (\lambda - \bar{\lambda}) X \bar{X} dx = 0 \tag{7.69}$$

hence $\lambda = \bar{\lambda}$. ∎

Theorem 20. *Assume for a symmetric boundary condition for $x \in [a,b]$. if $f(x)f'(x)\Big|_a^b \leq 0$ for all functions $f(x)$ satisfying the boundary conditions, then all eigenvalues are positive.*

Proof. Let $X(x)$ be the function satisfying the symmetric boundary conditions as well as the property above, and $-X''(x) = \lambda X(x)$. Then use Green's first identity, we have

$$\int_a^b X''(x) X(x) dx = X'(x) X(x)\Big|_a^b - \int_a^b (X'(x))^2 dx \tag{7.70}$$

which means

$$-\lambda \int_a^b X^2(x) dx = X'(x) X(x)\Big|_a^b - \int_a^b (X'(x))^2 dx \leq 0 \tag{7.71}$$

hence $\lambda \geq 0$. ∎

This also explains why earlier we state $\lambda > 0$ for the Dirichlet boundary condition for wave equation using separation of variables.

7.3 Convergence of Fourier Series

Definition 23. *A sequence of functions $\{f_n\}_{n=1}^{\infty}$ defined on (a,b) converges in L^2 to f if and only if*

$$\int_a^b |f_n(x) - f(x)|^2 dx \overset{n\to\infty}{\to} 0.$$

(7.72)

Theorem 21 (Bessel's Inequality). *Let $\varphi(x) \in L^2((a,b))$ and suppose $\{X_n(x)\}$ is an orthogonal family of functions on (a,b). Define*

$$A_n := \frac{\langle \varphi(x), X_n(x) \rangle}{||X_n(x)||^2},$$

(7.73)

then

$$\sum_{n=1}^{\infty} A_n^2 \int_a^b |X_n(x)|^2 dx \leq \int_a^b |\varphi(x)|^2 dx.$$

(7.74)

Proof. Let $\{c_n\}$ be a sequence of real numbers, define

$$E_N := \int_a^b \left| \varphi(x) - \sum_{n=1}^{N} c_n X_n(x) \right|^2 dx$$

(7.75)

$$= \int_a^b |\varphi(x)|^2 dx - 2 \sum_{n=1}^{N} c_n \int_a^b \varphi(x) X_n(x) dx + \sum_{m=1}^{N} \sum_{n=1}^{N} c_m c_n \int_a^b X_m(x) X_n(x) dx$$

(7.76)

$$= ||\varphi(x)||^2 - 2 \sum_{n=1}^{N} c_n \langle \varphi(x), X_n(x) \rangle + \sum_{n=1}^{N} c_n^2 ||X_n||^2 \quad \text{by orthogonality of } \{X_n\}$$

(7.77)

View above as a function of each c_n, we see that it is then a quadratic function: $f(c_n) = a_n c_n^2 + b_n c_n + d_n$ and each c_n is minimized at

$$c_n = -\frac{b_n}{2a_n} = \frac{\langle \varphi(x), X_n(x) \rangle}{||X_n(x)||^2}.$$

(7.78)

With c_n minimized as above, we have

$$E_N = ||\varphi(x)||^2 - \sum_{n=1}^{N} A_n^2 ||X_n(x)||^2.$$

(7.79)

By definition, $E_N \geq 0$, so

$$||\varphi(x)||^2 \geq \sum_{n=1}^{N} A_n^2 ||X_n(x)||^2$$

(7.80)

which is,

$$\sum_{n=1}^{N} A_n^2 \int_a^b |X_n(x)|^2 dx \leq \int_a^b |\varphi(x)|^2 dx$$

(7.81)

and the Bessel's inequaity is proven by taking $N \to \infty$. ∎

Theorem 22 (Parseval's Equality). *Let $\varphi(x) \in L^2((a,b))$, and $X_n(x), A_n(x)$ be the eigenfunctions and Fourier coefficients of any general Fourier series for $\varphi(x)$ respectively. Then*

$$\sum_{n=1}^{\infty} A_n^2 \int_a^b |X_n(x)|^2 dx = \int_a^b |\varphi(x)|^2 dx.$$

(7.82)

We will not prove this theorem.

Let $\varphi(x) \equiv 1 \in L^2((0, \pi))$, using classical Fourier sine series defined on $(0, \pi)$, we have

$$X_n(x) = \sin\frac{n\pi x}{l} = \sin(nx) \tag{7.83}$$

and

$$A_n = \frac{\langle \varphi(x), X_n(x) \rangle}{||X_n(x)||^2} = \frac{\int_0^\pi \sin(nx)dx}{\int_0^\pi \sin^2(nx)dx} = \frac{4}{n\pi} \quad \text{for odd } n. \tag{7.84}$$

hence using Parseval's equality, we have

$$\sum_{\text{odd } n} \left(\frac{4}{n\pi}\right)^2 \cdot \int_0^\pi \sin^2(nx)dx = \int_0^\pi 1^2 dx \tag{7.85}$$

which is

$$\sum_{\text{odd } n} \left(\frac{4}{n\pi}\right)^2 \cdot \frac{\pi}{2} = \pi \tag{7.86}$$

i.e,

$$\sum_{\text{odd } n} \frac{1}{n^2} = \frac{\pi^2}{8}. \tag{7.87}$$

Theorem 23 (Riemann-Lebesgue Lemma). *If $\varphi(x) \in L^2((a,b))$, then the classical Fourier coefficients of $\varphi(x)$ goes to 0 as $n \to \infty$.*

By Riemann-Lebesgue lemma, recall the Dirichlet boundary conditions on $[0,l]$, we have that every $\varphi(x)$ defined on $(0,l)$ with $||\varphi(x)||^2 < \infty$ satisfies

$$\int_0^l \varphi(x)\sin\frac{n\pi x}{l}dx \to 0, \quad \text{as } n \to \infty. \tag{7.88}$$

Since C_C^∞ functions are surely square integrabe, so the above equation also means

$$\sin\frac{n\pi x}{l} \to 0 \quad \text{in distribution.} \tag{7.89}$$

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