McGill Fall 2024 - MATH466 - Honors Complex Analysis
Taught By: Prof. Jacques Claude Hurtubise
Notes By: Megan Millet

Contents

1 Introduction to Complex Numbers

1.1 Review of Complex Numbers

$\mathbb{C}$ is the set of complex numbers, which is defined as $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$. Here, a is the real part of the complex number, and b is the imaginary part. The imaginary unit i is defined as $i^2 = -1$. We can also write a complex number in polar form as $z = r(\cos(\theta) + i\sin(\theta))$, where r is the magnitude of the complex number and θ is the argument of the complex number. The magnitude of a complex number is defined as $|z| = \sqrt{a^2 + b^2}$, and the argument of a complex number is defined as $\arg(z) = \theta = \arctan(\frac{b}{a})$. We can also write a complex number in exponential form as $z = re^{i\theta}$.

The complex conjugate of a complex number $z = a+bi$ is defined as $\bar{z} = a-bi$. The complex conjugate of a complex number has the following properties:

- $\overline{z + w} = \bar{z} + \bar{w}$
- $\overline{zw} = \bar{z}\bar{w}$
- $\overline{z} = z$ if and only if z is real
- $z\bar{z} = |z|^2$

We can perform arithmetic operations on complex numbers. For example, we can add, subtract, multiply, and divide complex numbers. The following are the formulas for these operations:

- Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
- Subtraction: $(a + bi) - (c + di) = (a - c) + (b - d)i$
- Multiplication: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
- Division: $\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$

The Cauchy-Schwarz inequality states that for any two complex numbers z and w, $|z \cdot w| \leq |z| \cdot |w|$. This inequality is a generalization of the triangle inequality, which states that for any two complex numbers z and w, $|z + w| \leq |z| + |w|$.

1.2 Real vs. Complex Derivatives and Cauchy Riemann Equations

Let V be a real, 2-dimensional vector space. We can turn it into a complex 1-dimensional vector space by adding an $\mathbb{R}$ linear map $I : V \to V$ with $I^2 = -\mathbb{1}$. Choose a $v_1 \in V$, take v_2 a vector perpendicular to v_1, then these two vectors form an oriented basis.

Definition 1.1. (Complex Derivative) Let $f = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathbb{C}^1$ function. It has a **complex derivative** at $(x_0, y_0) \iff$ differential commutes with $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ In other words, $Df \cdot I - I \cdot Df = 0$

Definition 1.2. (Cauchy Riemann Equations) The **Cauchy Riemann Equations** are the following:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We use this to motivate $\mathbb{C}$-derivatives by taking $f : \mathbb{C} \to \mathbb{C}$, for $f = u + iv$, to another function from $\mathbb{C} \to \mathbb{C}$. Consider $z = x + iy$. We have that

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

Observe that:

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}} \\ &= \frac{1}{2}(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}) + \frac{i}{2}(\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}) \\ &= \frac{1}{2}[(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) + i(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})] \\ &= 0 \iff \text{Cauchy Riemann for } f = u + iv\end{aligned}$$

On the other hand, look at $f = u + iv$:

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{1}{2}(\frac{\partial(u+iv)}{\partial x} - i\frac{\partial(u+iv)}{\partial y}) \\ &= \frac{1}{2}[(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + i(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})]\end{aligned}$$

$$\frac{\partial f}{\partial z}(a+bi) = \frac{a}{2}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) - \frac{b}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) + \frac{ai}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) + \frac{b}{2}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

If $\frac{\partial f}{\partial \bar{z}} = 0$ (i.e. Cauchy Riemann) then

$$\frac{\partial f}{\partial z}(a+ib) = (a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + i[(a\frac{\partial v}{\partial x} + b\frac{\partial v}{\partial y})])$$

But Df$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \implies \frac{\partial f}{\partial z}$ is the complex version of Df.

Let $f \in \mathbb{C}^1$. f is **holomorphic** $\iff$ it has a complex derivative on its domain of definition
$\iff \frac{\partial f}{\partial \bar{z}} = 0$ on its definition
$\iff f$ satisfies Cauchy Riemann equations when written as $f = u + iv$

If f is holomorphic, then $Df(a+ib) = \frac{\partial f}{\partial z}(a+ib)$.
Some Examples of the Properties of Holomorphic Functions:

- $\frac{\partial}{\partial z}(z) = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(x+iy) = \frac{1}{2} + \frac{1}{2} = 1$
- Satisfies Liebnitz Rules: $\frac{\partial}{\partial z}(fg) = f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}$
- $\frac{\partial}{\partial z}(\bar{z}) = 0$
- Satisfies Chain Rule: $\frac{\partial}{\partial z}(f(g(z))) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial z}$

Remark: $[D, I] \iff D = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

That is, a matrix D commutes with I $\iff$ D is a multiple of the identity matrix and the matrix I.
We can see this in two steps:

1. Compute $DI - ID = 0$
2. Eigenvalues of D are $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$.

Example 1.1. a=cos(θ), b=sin(θ) $\implies$ D is a rotation matrix of $e^{i\theta}$.

2 Integration

2.1 Green's Theorem

Recall: line integrals in $\mathbb{R}^2$, given

1. parametrized piecewise C^1 cuve $C : [a, b] \to \mathbb{R}^2$
2. an expression $\rho : f(x, y)dx + g(x, y)dy$ for f,g continuous functions

Definition 2.1. (Line Integral of ρ) We define the line integral of ρ along C as:

$$\int_C \rho = \int_a^b (f(x(t), y(t))\frac{dx}{dt} + g(x(t), y(t))\frac{dy}{dt})dt$$

Key points:

- The line integral is independent of the parametrization of C (by the Chain Rule)
- The line integral is additive over the curve
- The line integral is linear over the integrand

Theorem 2.1. (Green's Theorem) $\int_C \rho = \int_D d\rho$ where it is understood that $C : [a, b] \to \mathbb{R}$ and D is an open set in $\mathbb{R}^2$ with boundary C.

Proof. Case 1: D is a rectangle (lined up with the coordinate axes)
We integrate $fdx + gdy$ on ∂D

$$\int_C fdx + gdy = \int_{a_1}^{a_2} f(x, b_1)dx + \int_{b_1}^{b_2} g(a_2, y)dy - \int_{a_1}^{a_2} f(x, b_2)dx - \int_{b_1}^{b_2} g(a_1, y)dy$$

Now by Fubini's Theorem,

$$\int_{a_1}^{a_2} dx \int_{b_1}^{b_2} dy(-\frac{df}{dy} + \frac{dg}{dx}) = \int_{a_1}^{a_2} dx(-f(x, y))\Big|_{y=b_1}^{y=b_2} + \int_{b_1}^{b_2} dy(g(x, y))\Big|_{x=a_1}^{x=a_2}$$

$$= \int_{a_1}^{a_2} (-f(x, b_2) + f(x_{b_1}))dx + \int_{b_1}^{b_2} (g(a_2, y) - g(a_1, y))dy$$

Case 2: Suppose D is not a rectangle
We can either cut D into an infinite family of rectangles and sum them up, or we can do a change of variables.
Suppose you have $D \overset{M}{\simeq}$ rectangle and $\partial D = C \simeq \partial$rectangle; M is an invertible C^1 map with C^1 inverse.
Now we can use:

$$\int_C fdx + gdy = \int_{\partial D_R} \mu^*(fdx + gdy)$$

$$\int_D \frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} dxdy = \int_{D_R} \mu^*(-\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x})dxdy$$

and μ^* commutes with d

$$\mu^*(f(x,y)dx + g(x,y)dy) = (f\frac{\partial x}{\partial \hat{x}} + g\frac{\partial y}{\partial \hat{x}})d\hat{x} + (f\frac{\partial x}{\partial \hat{y}} + g\frac{\partial y}{\partial \hat{y}})d\hat{y}$$

And then we can check d of this:

$$-\frac{\partial f}{\partial \hat{y}}\frac{\partial x}{\partial \hat{x}} - f\frac{\partial^2 x}{\partial \hat{x}\partial \hat{y}} - \frac{\partial g}{\partial \hat{y}}\frac{\partial y}{\partial \hat{x}} - g\frac{\partial^2 y}{\partial \hat{x}\partial \hat{y}} + \frac{\partial f}{\partial \hat{x}}\frac{\partial x}{\partial \hat{y}} + f\frac{\partial^2 x}{\partial \hat{x}\partial \hat{y}} + \frac{\partial g}{\partial \hat{x}}\frac{\partial y}{\partial \hat{y}} + g\frac{\partial^2 y}{\partial \hat{x}\partial \hat{y}} \tag{1}$$

And:

$$\mu^*(-\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x})dxdy = (-\frac{\partial f}{\partial \hat{y}}\frac{\partial x}{\partial \hat{x}} - \frac{\partial f \partial y^2}{\partial \hat{y}\partial \hat{x}} + \frac{\partial g}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial x} + \frac{\partial g}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial x})\det\begin{pmatrix} \frac{\partial x}{\partial \hat{x}} & \frac{\partial y}{\partial \hat{x}} \\ \frac{\partial x}{\partial \hat{y}} & \frac{\partial y}{\partial \hat{y}} \end{pmatrix} \tag{2}$$

We can compare the terms (1) and (2) and see that $\mu^* \circ d = d \circ \mu^*$ and the invariance of the integrals, $\implies$ Green for D. □

2.2 Complex Line Integrals

Definition 2.2. Define a $\mathbb{C}$-line integral by: Let $z = x + iy$, $f(z) = u(z) + iv(z)$ then

$$\int_C f(z)dz \equiv \int_C (u+iv)d(x+iy) = \int_C (udx - vdy) + i(vdx + udy) = \int_C udx - vdy + i\int_C vdx + udy$$

And by applying d to $f(z)dz$:

$$d(udx - vdy) = (-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x})dxdy$$

$$d(vdx + udy) = (-\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x})dxdy$$

If f is holomorphic then it satisfies the Cauchy-Riemann equations, so $d(f(z)dz) = 0$.

Theorem 2.2. (Cauchy's Theorem) Can apply Green for both real and imaginary parts.

$$\int_{C'} f(z)dz - \int_C f(z)dz = \int_D d(f(z)dz) = 0$$

If $C' \cup (-C)$ bound a disk or something equivalent and $f(z)$ is defined and holomorphic on D (where D represents the area between the curves).

If $f(z)$ is holomorphic on a disk D we can define an anti-derivative by $g(z) = \int_{z_0}^{z} f(s)ds$

How to check that $\frac{\partial g}{\partial z} = f$?
$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$

1. Integrate $\int f(s)ds$ on a path
2. Integrate on another path, gives the same result

2.3 Cauchy Integral Formula

Let C, C' be paths from $z_0 \to z$. Then they bound a disk together on which f is holomorphic. Then:

$$\int_C f(z) = \int_{C'} f(z)dz \implies g(z) = \int_{z_0}^{z} f'(s)ds$$

is well-defined.
More general equivalence of paths: We have two paths $C, C' : [a,b] \to U \subset \mathbb{R}^2$ are holotopic $\iff$ there exists $F : [a,b] \times [0,1] \to U$ open.
Some properties:

(i) $F|_{[a,b]\times\{0\}} = C$

(ii) $F|_{[a,b]\times\{1\}} = C'$

(iii) $F|_{\{a\}\times[0,1]} \equiv C(a) = C'(a)$

(iv) $F|_{\{b\}\times[0,1]} \equiv C(b) = C'(b)$

Proposition 2.1. Suppose $f(z)$ is holomorphic on U, and $C, C' : [a,b] \to U$ are holotopic in U. Then $\int_C f(z)dz = \int_{C'} f(z)dz$.

Proof. (i) We first reduce the problem to Greens

(ii) Then we pullback $f(z)dz$ to $[a,b] \times [0,1]$ using F and use Green there.

□

Exercise 2.1. Exponentials e^z are the holomorphic solution to $f'(z) = f(z)$ and $f(0) = 1$. We first easily see that $e^z = \frac{d}{dz}e^z = e^z$. We now want to show that e^z is holomorphic. This is trivial as e^z is differentiable everywhere on the complex plane.

Theorem 2.3. (Cauchy's Integral Formula) Let $f : U \to R^2$ be holomorphic. Let γ be the boundary of the disk $D(z_0, r)$, and $D(\bar{z_0}, r) \subset U$. Then:

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

Proof. By using the Cauchy integral theorem, one can show that the integral over C (or the closed rectifiable curve) is equal to the same integral taken over an arbitrarily small circle around a. Since $f(z)$ is continuous, we can choose a circle small enough on which $f(z)$ is arbitrarily close to $f(a)$. On the other hand, the integral

$$\oint_C \frac{1}{z-a}\, dz = 2\pi i,$$

over any circle C centered at a. This can be calculated directly via a parametrization (integration by substitution) $z(t) = a + \varepsilon e^{it}$, where $0 \le t \le 2\pi$ and ε is the radius of the circle.

Letting $\varepsilon \to 0$ gives the desired estimate:

$$\left|\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\, dz - f(a)\right| = \left|\frac{1}{2\pi i}\oint_C \frac{f(z)-f(a)}{z-a}\, dz\right| = \left|\frac{1}{2\pi i}\int_0^{2\pi} \left(\frac{f(z(t))-f(a)}{\varepsilon e^{it}} \cdot \varepsilon e^{it} i\right) dt\right|$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(z(t))-f(a)|}{\varepsilon}\, \varepsilon\, dt \le \max_{|z-a|=\varepsilon} |f(z)-f(a)| \xrightarrow[\varepsilon\to 0]{} 0.$$

□

As a consequence,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} d$$
$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$
$$f''(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^3} d\zeta$$

It is infinitely differentiable and analytic.

3 Series Expansions

Equivalent notions for U open in $\mathbb{C}$:

(i) U is diffeomorphic to a disk $D : \{|z_0| < r\}$

(ii) U is diffeomorphic to a rectangle

(iii) U is contractible and there exists a homotopy $F : U \times [0,1] \to U$

$F|_{U\times[0,1]} =$ identity

$F|_{U\times\{1\}} =$ point

(iv) U is simply connected, that is for all closed loops in U, called $l : S^1 \to U$, there exists $L : S^1 \times [0,1] \to U$

$L|_{S^1\times\{0\}} = l$

$L|_{S^1\times\{1\}} =$ point

Things that imply a disk:

- convexity
- star-shapedness

These notions gave us:

1. Calculus over $\mathbb{C}$
2. Cauchy's Theorem

Application of Cauchy's Theorem:

(i) Differentiability: $f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)^2} d\zeta$

(ii) Analytic

Recall: Root test for convergence radius of power series

1. Start from $\frac{1}{1-a\zeta} = 1 + a\zeta + a^2\zeta^2 + \dots, \quad a \in \mathbb{R}^2$ which converges when $|\zeta| < \frac{1}{a}$
2. For some $\sum b_n\zeta^n$, we compare it to $\sum_{n=0}^{\infty} a^n\zeta^n$. If $|b_n|^{1/n} < a + \epsilon, n > n_0$ then $|b_n| < |a+\epsilon|^n, n > n_0$. Hence if $|\zeta| < \frac{1}{a+\epsilon}$ this converges absolutely.

3.1 Power Series Expansions

From Cauchy's Formula we have:

- f is ∞-differentiable
- Cauchy estimate
- And that f is analytic

Theorem 3.1. (Abel's Theorem) Suppose $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges at z_1. Then it converges at z for $|z-z_0| < |z_1 - z_0|$.

Proof. Convergence implies that $|a_k(z-z_0)^k| < M$ for $k > k_0$. Then $\sum_{k=0}^{\infty} |a_k||(z-z_0)^k| \leq \sum_{k=k_0}^{\infty} M\frac{|z-z_0|^k}{|z_1-z_0|^k}$. And this is a geometric series which converges as $\frac{|z-z_0|^k}{|z_1-z_0|^k} < 1$. $\square$

Remark 3.1. Convergence is uniform for $|z-z_0| < c|z_1-z_0|$ for $c < 1$.

3.2 Liouville's Theorem

Theorem 3.2. (Liouville's Theorem) f is bounded and entire (holomorphic on all of $\mathbb{C}$) $\implies$ f is constant.

Proof. We can see this by $\frac{\partial f}{\partial z}(z_0) \leq \frac{M(r)}{r}$ and if $M(r) < c \quad \forall r \implies \frac{\partial f}{\partial z} = 0$ $\square$

Theorem 3.3. (Alternate Liouville's Theorem) $f(k)$ is entire and $|f(z)| \leq cr^k \implies f$ is a polynomial of degree at most k.

Theorem 3.4. (Fundamental Theorem of Algebra) Any monic polynomial $p(z) = z^k + a_1z^{k-1} + \ldots + a_k, k > 0$ has a root ($\exists z_0$ such that $p(z_0) = 0$).

Proof. Suppose this is not the case. Then $f(z) = \frac{1}{p(z)}$ is entire and bounded. By Liouville, f is constant, but this is a contradiction. $\square$

3.3 Limits of Holomorphic Functions

Theorem 3.5. Let $f_j : U \to \mathbb{C}$ be holomorphic. If $f_j \to f$ uniformly on all compact subsets of U, then f is holomorphic.

Proof. $f(z) = \lim f_j(z) = \lim \frac{1}{2\pi i}\oint_C \frac{f_j(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta-z}d\zeta$ $\square$

Theorem 3.6. Let $f_j : U \to \mathbb{C}$ be holomorphic. If $f_j \to f$ on any compact set of U, then $\frac{\partial^k f_j}{\partial z^k} \to \frac{\partial^k f}{\partial z^k}$

Proof. f holomorphic $\implies$ f is infinitely differentiable. Then we can apply the same argument as above. $\square$

Proposition 3.1. Take $f : U \to \mathbb{C}$. Suppose $f(z_i) = 0, \quad j = 1, 2, \ldots$ and $z_j \to z_0, z_0 \in U$ and z_0 different from all z_j. Then $f \equiv 0$ on the component of U containing z_0.

Proof. Suppose toward a contradiction. $f(z_0) = 0$ by continuity. $f(z) = \sum_{n\geq 1} a_n(z-z_0)^n$. Let a_{n_0} be the first nonzero coefficient. We can rewrite this as:

$$f(z) = a_{n_0}(z-z_0)^{n_0}(1 + \sum_{n\geq 1} a_{n_0+n}(z-z_0)^n)$$

But then we can rewrite $f_j(z) = (z_j - z_0)^{n_0}(a_{n_0} + a_{n_0+1}(z_j - z_0) + \ldots)$, but note that a_{n_0} and $(z_j - z_0)^{n_0}$ are both nonzero, hence $f_j(z) \neq 0$ a contradiction. $\square$

Theorem 3.7. (Morera's Theorem) Let $f \in C^{\infty}(U)$. If for all closed paths C are in U and $\int_C f(z)dz = 0$ then f is holomorphic.

Proof. At $p \in U$, choose a disk $\hat{U}$ with $p \in \hat{U} \subset U$. Then we can define $F(z) = \int_p^z f(\zeta)d\zeta$. We can then apply the Fundamental Theorem of Calculus to get $F'(z) = f(z)$. $\square$

Lemma 3.1. Suppose U is an open subset of $\mathbb{C}$ and that $f : U \to \mathbb{C}$ is holomorphic, with $f'(z_0) \neq 0$. Then f has a local inverse at z_0. That is, there exists $\hat{U}$ open such that $z_0 \in \hat{U} \subset U$, $\hat{V}$ open such that $f(z_0) \in \hat{V}$ with $\hat{f} : \hat{U} \to \hat{V}$ a holomorphic isomorphism.

Proof. If $f'(z_0) \neq 0$ then $Df(z_0)$ is invertible. Then by the inverse function theorem, f has a local C^{∞} inverse h. Then h is holomorphic by Morera's Theorem. $\square$

Lemma 3.2. (There exists Nth roots) Near $z_0 \neq 0$, $z \mapsto z^n$ has an inverse.

Proof.

1) $\frac{d(z^n)}{z} \neq 0$ at z_0, so we can apply the inverse function theorem.

2) Or we can solve $[\sum_{k=0}^{\infty} a_k(z - z_0)^k]^N = z = z_0 + (z - z_0)$

3) We can consider polar coordinates and solve for r and θ.

$\square$

Corollary 3.1. (Open Mapping Theorem) Let U be an open subset and $f : U \to f(U) \subset \mathbb{C}$ for holomorphic, non-constant function f. Then $f(U)$ is open.

Corollary 3.2. (Maximum Principle) Holomorphic, non-constant functions on U attain their maximum norm (if at all) on the boundary.

Proof. Suppose not. We let $|f(z_0)| \geq |f(z)|, \forall z \in U$, with $f(V)$ open, $f(z_0) \subset f(V)$ open. Then there exists a point in $f(V)$ further from the origin than $f(z_0)$, hence a contradiction. $\square$

We can expand this by supposing that $p \in U$ open, $f : U \backslash \{p\} \to \mathbb{C}$ is holomorphic.
Question: what can go bad at p?

(1) f is bounded on a neighborhood of p, $|f(z)| < M \implies$ Riemann removable singularity theorem (f is holomorphic at p)

(2) $|f| \to \infty$ on a neighborhood of $p \implies$ Riemann on $\frac{1}{f} \implies \frac{1}{f}$ holomorphic (poles property), Casorati-Weierstrass

(3) Neither $\implies$ essential singularity

Example 3.1. Essential singularity: $f(z) = e^{\frac{1}{z}}$ at $z = 0$.

Theorem 3.8. ((3) Casorati-Weierstrass) Let $f : D(p, r_0)\backslash\{p\} \to \mathbb{C}$ be holomorphic. Let there be an essential singularity at p. Then $f(D(p, r_0)\backslash\{p\})$ is dense in $\mathbb{C}$.

Proof. Suppose not. Then $\exists \lambda \in \mathbb{C}$, a disk $D(\lambda, \epsilon)$ with $f(D(p, r)\backslash\{p\} \cap D(\lambda, \epsilon)) = \emptyset$. Now consider $|g(z)| = |\frac{1}{f(z) - \lambda}| < \frac{1}{\epsilon} \implies$ essential singularities, and $g(z)$ is holomorphic at p. But then consider $f(z) = \lambda + \frac{1}{g(z)}$. Then there are two cases:

1. $g(p) \neq 0 \implies f$ at holomorphic at p (essential singularity)

2. $g(p) = 0 \implies |f(z)| \to \infty$ at p (pole)

This is a contradiction as we assumed p was an essential singularity. $\square$

Theorem 3.9. ((1) Riemann Removable Singularity Theorem) Let $f : D(p, r_0)\backslash\{p\} \to \mathbb{C}$ be holomorphic, f is bounded on $D(p,r)$ so $|f(z)| \leq M$, then:

(i) $\lim_{z\to p} f(z)$ exists

(ii) It extends f holomorphically at p

Proof. Set $g(z) = (z-p)^2 f(z)$. Then since f is bounded, $|g| \leq M|z-p|^2$. We will now show that g is in C^1:

$$\begin{aligned}\frac{g(z)-g(p)}{z-p} &= \frac{(z-p)^2 f(z) - p^2 f(p)}{z-p} \\ &= (z-p)f(z) + p^2\frac{f(z)-f(p)}{z-p} \\ &\to 0\end{aligned}$$

□

Proof. ((2) f has a pole at p) $\exists D(p,r)$ with $|f|_{D(p,r)} > M \implies \frac{1}{f}$ holomorphic on $D(p,r)\backslash\{p\}$ so $|\frac{1}{f}| < \frac{1}{M}$ $\implies$ holomorphic at p by Riemann removable singularity theorem. But since it is holomorphic at p, we can rewrite it as a power series:

$$\frac{1}{f(z)} = \sum_{n\geq 0} a_n(z-p)^n \implies f(z) = \sum_{i=-k}^{\infty} f_i(z-p)^i$$

□

Definition 3.1. We say that $\sum\limits_{i=-\infty}^{\infty} a_i(z-p)^i$ converges absolutely at $z \iff \sum\limits_{i=0}^{\infty} a_i(z-p)^i$ and $\sum\limits_{-\infty}^{-1} a_i(z-p)^i$ converges absolutely at z.

Theorem 3.10. Suppose that $\sum\limits_{i=-\infty}^{\infty} a_i(z-p)^i$ converges absolutely at z_0. Then $\exists$ an annulus of convergence $r_1 < |z-p| < r_2$ containing z_0.

Proof. (1) $\sum\limits_{i=0}^{\infty} a_i(z_0-p)^i$ converges $\implies$ Abel's theorem so $\sum\limits_{i=0}^{\infty} a_i(z-p)^i$ converges for $|z-p| < r_2$ with $|z_0 - p| < r_2$.

(2) $\sum\limits_{-\infty}^{-1} a_i(z_0-p)^i = \sum\limits_{i=1}^{\infty} a_{-i}\frac{1}{(z_0-p)^i}$ converges $\implies$ Abel's theorem so $\sum\limits_{i=1}^{\infty} a_{-1}(z-p)^i$ converges for $|z-p| < \frac{1}{r_1}$ and $|\frac{1}{z_0-p}| < \frac{1}{r_1} \iff |z_0-p| > r_1$.

□

4 Meromorphic Functions and Residues

4.1 Laurent Expansions

Theorem 4.1. Let f be holomorphic on $r_1 < |z-p| < r_2$. Then f has a Laurent Series $\sum\limits_{i=-\infty}^{\infty} a_i(z-p)^i$ such that $a_i = \oint_{C'} \frac{f(\zeta)}{(\zeta-p)^{i+1}}d\zeta$ where C' is any circle in the annulus.

Proof. We have that:

$$\begin{aligned}
f(z) &= \frac{-1}{2\pi i}\int_{C_{s_1}} \frac{f(\zeta)}{\zeta - p} d\zeta + \frac{1}{2\pi i}\int_{C_{s_2}} \frac{f(\zeta)}{\zeta - p} d\zeta \\
&= \frac{1}{2\pi i}\int_{C_{s_1}} \frac{f(\zeta)}{1 - \frac{\zeta - p}{z - p}} \frac{d\zeta}{z - p} + \frac{1}{2\pi i}\int_{C_{s_2}} \frac{f(\zeta)}{1 - \frac{z-p}{\zeta - p}} \frac{d\zeta}{\zeta - p} \\
&= \frac{1}{2\pi i}\int_{C_{s_1}} f(s)(1 + \frac{\zeta - p}{z - p} + \frac{(\zeta - p)^2}{(z-p)^2}) \frac{d\zeta}{z - p} + \frac{1}{2\pi i}\int_{C_{s_2}} f(s)(1 + \frac{z - p}{\zeta - p} + \frac{(z-p)^2}{(\zeta - p)^2}) \frac{d\zeta}{\zeta - p} \\
&= \frac{1}{2\pi i}\sum_{i=1}^{\infty} \frac{1}{(z-p)^i}\int_{C_{s_1}} f(\zeta)(\zeta - p)^{i-1} d\zeta + \frac{1}{2\pi i}\sum_{i=1}^{\infty} \frac{1}{(z-p)^{-i}}\int_{C_{s_2}} f(\zeta)(\zeta - p)^{-i-1} d\zeta \\
&= \frac{1}{2\pi i}\sum_{j=-\infty}^{\infty} (z-p)^j (\int_{C_r} \frac{f(\zeta)}{(\zeta - p)^{j+1}} d\zeta)
\end{aligned}$$

By change of variables $i = -j$ and the smoothness of the integrals over C_{s_1} and C_{s_2} are equivalent to over C_r. □

Proposition 4.1. If f is holomorphic on $D^*_{(p,r)} = D_{(p,r)} \setminus \{p\}$ then f has a Laurent Series $\sum_{i=-\infty}^{\infty} a_i(z-p)^i$ on $D^*_{(p,r)}$.

4.2 Calculus of Residues

Definition 4.1. (Meromorphic) A function f is said to be meromorphic if it has a pole at $p \iff a_i = 0 \quad \forall i < -k$ where k is the order of the pole.

Definition 4.2. (Residue) Let f be holomorphic on $D^*_{(p,r)}$. We call the reside of f at p the -1 coefficient of the Laurent Series.

Example 4.1. $\text{Res}_0(\frac{1}{z}) = \frac{1}{2\pi i} log(z)|_{C(0)}^{C(1)} = 1$

Example 4.2. $\text{Res}_0(\frac{sin(z)}{z}) = \text{Res}_0(\frac{z - \frac{z^3}{3!} + \frac{z^5}{5! - \dots}}{z}) = 0$

Example 4.3. $\text{Res}_0(\frac{cos(z)}{z}) = \text{Res}_0(\frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots}{z}) = 1$

Proposition 4.2. If f is holomorphic at $D^*_{(p,r)}$ and meromorphic at p then $f(z) = \sum_{i=-k}^{\infty} a_i(z-p)^i$

Definition 4.3. (Principle Part) We define the principle part of f at p to be:

$$\text{PP}_p(f) = \sum_{i=-k}^{-1} a_i(z-p)^i$$

Example 4.4. $\text{PP}_1(\frac{sin(z)}{z-1}) = \text{PP}_1(\frac{(\sin(1)) + \cos(1)(z-1) + \dots}{z-1}) = \frac{sin(1)}{z-1}$

Definition 4.4. (Rational Function) A rational function on $\mathbb{C}$ is the quotient of 2 polynomials, such that $\frac{p(z)}{q(z)}$ coprime and has poles at the zeroes of $q(z)$.

Definition 4.5. (Division algorithm) Suppose $\deg(p) \geq \deg(q)$, then the division algorithm is $p = aq + r$ for a, r polynomials with $\deg(r) < \deg(q)$.

Proposition 4.3. If r, q are polynomials such that $\deg(r) < \deg(q)$, then

$$\frac{r}{q} = \sum_{\text{zeroes of q denoted } z_i} \text{PP}_{z_i}(\frac{r}{q})$$

Proof. At z_i we have that $\frac{r}{q} = \text{PP}_{z_i}(\frac{r}{q}) + a_0 + a_1(z - z_i) + a_2(z - z_i)^2 + \ldots$. Then $\frac{r}{q} - \text{PP}_{z_i}(\frac{r}{q})$ is holomorphic and bounded at z_i. But also, the sum over the PP_{z_i} would be holomorphic and bounded at z_i. This implies that $\frac{r}{q} - \sum_j \text{PP}_{z_i}(\frac{r}{q})$ is bounded on R.
On the other hand, as $R \to \infty$,

$$|\text{PP}_{z_i}(\frac{r}{q})| = |\frac{a_1}{z - z_i} + \frac{a_2}{(z - z_i)^2} + \ldots| \to 0$$

Thus since $\deg(r) < \deg(q)$, we have that $|\frac{r}{q}| \to 0$ as $R \to \infty$. This implies Liouville's theorem and that $\frac{r}{q}$ is a constant, which in fact must be 0 as $R \to \infty$. □

4.3 Applications of Residues

Riemann-Sphere: $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$ (Check lecture 9 notes for calculations)

(1) As a manifold: Take 2 copies of $\mathbb{C}$ and glue them together at ∞.

(2) Stereographic projection

(3) As the projective line

What are holomorphic functions on $\mathbb{P}^1$?

Definition 4.6. f is holomorphic at $z_0 \iff f$ is holomorphic as a function $\mathbb{C}_0 \to \mathbb{C}$ if $z_0 \in \mathbb{C}_0$ or $\mathbb{C}_\infty \to \mathbb{C}$ if $z_0 \in \mathbb{C}_\infty$.

Proposition 4.4. $\mathbb{P}^1$ is compact $\implies f$ is constant.

Definition 4.7. $\forall U \subset \mathbb{C}$ where U is open, a meromorphic function on U (only poles in U has singularities) $\equiv$ holomorphic function $f : U \to \mathbb{P}^1(\mathbb{C})$

Theorem 4.2. All meromorphic maps $\mathbb{P}^1 \to \mathbb{P}^1$ are rational functions.

Proof. $\mathbb{P}^1$ compact and zeroes have no cumulation points then there is a finite number of zeroes. But then pole of f in $\mathbb{C}_0$ label them $p_1, \ldots, p_k$ and possibly ∞ as an extra pole. At p_i let m_i be the multiplicity of the pole. On $\mathbb{C}_0$ look at $f \prod_{i=1}^k (z - p_i)^{m_i}$ this is bounded at the $p_i's$. Also, it is entire. Then $f \prod_{i=1}^k (z - p_i)^{m_i}$ has a pole at ∞, growth bounded by $|z|^m$. Then this is a polynomial and $f \prod_{i=1}^k (z - p_i)^{m_i} = \prod_{j=1}^l (z - a_i)c \implies f = \frac{\prod_{i=1}^l (z - a_i)c}{\prod_{i=1}^k (z - p_i)^{m_i}}$ □

Theorem 4.3. (Residue Theorem)
Let U be open in $\mathbb{C}$, $\bar{U}$ compact, $\partial \bar{U}$ piecewise smooth, and $f : \bar{U} \setminus \{p_1, \ldots, p_n\} \to \mathbb{C}$ holomorphic. Then:

$$\frac{1}{2\pi i} \int_{\partial U} f(z) dz = \sum_{i=1}^n \text{Res}_{p_i}(f)$$

Proof. Set $V = U \setminus \bigcup_{i=1}^n D(p_i, r)$ such that the $D(p_i, r)$ are mutually disjoint. Then f is holomorphic on V. Then $\frac{1}{2\pi i} \int_{\partial V} f(z) dz = 0 = \int_{\partial U} f(z) dz - \sum_{i=1}^n \int_{D(p_i, r)} f(z) dz = \sum_{i=1}^n Res_{p_i}(f)$ □

Example 4.5. We want to integrate $\frac{z^2 - 4z + 2}{z(z-1)(z-2)}$ on $\partial D(0, 3)$. If we compute the residues at each simple pole, this integral evaluates to $2\pi i$.

Remark 4.1. If $f(z)$ has a simple pole at z_0, then $f(z) = \frac{1}{z-z_0}(\frac{g(z_0)}{g(z_0)+(z-z_0)g'(z_0)+\ldots})$. Then $\text{Res}_{z_0}(f) = g(z_0)$.

Remark 4.2. If $f(z)$ has a higher order pole at z_0, then $f(z) = a_{-k}(z-z_0)^{-k} + a_{-k+1}(z-z_0)^{-k+1} + \ldots + a_{-1}(z-z_0)^{-1} + \ldots$.
We want: $a_{-1}(z-z_0)^{-1}$. So we get it by multiplying $(z-z_0)^k f(z) = a_{-k} + a_{-k+1}(z-z_0) + \ldots + a_{-1}(z-z_0)^{k-1} + \ldots$
We now take the derivative:

$$\frac{d^{k-1}}{dz^{k-1}}((z-z_0)^k f(z))|_{z=z_0} = (k-1)!a_{-1} + (z-z_0)a_0 + \ldots$$

$$\implies \text{Res}_{z_0}(f) = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}((z-z_0)^k f(z))|_{z=z_0}$$

4.4 Calculus of Residues

We can use the residue theorem $\frac{1}{2\pi i}\int_C f(z)dz = \sum \text{Res}$ to do two of the following operations:

(1) To compute a definite integral (typically on a part of C)

(2) To compute a sum (can be an infinite sum) as a sum of residues

See lecture 10 notes for examples of (1). See lecture 11 notes for examples of (2).

Example 4.6. We want to compute $\int_{\mathbb{R}} \frac{1}{1+x^4}dx$. We will consider where the denominator has zeroes: $x = e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}$. We now compute using the complex version of the function:

$$\frac{1}{z^4+1} = \frac{1}{(z-e^{\pi i/4})(z-e^{3\pi i/4})(z-e^{5\pi i/4})(z-e^{7\pi i/4})}$$

Consider the case where $z = e^{\pi i/4}$:

$$\frac{1}{z-e^{\pi i/4}}(\frac{1}{(z-e^{3\pi i/4})(z-e^{5\pi i/4})(z-e^{7\pi i/4})} + O(z-e^{\pi i/4}))|_{z=e^{\pi i/4}}$$

We get the residue from this computation: $\text{Res}_{e^{\pi i/4}} = \frac{-1-i}{4\sqrt{2}}$.
We compute the other pole similarily: $\text{Res}_{e^{3\pi i/4}} = \frac{1-i}{4\sqrt{2}}$.
Now we need to look at the contour:
We have $\int_{-R}^{R} \frac{1}{1+x^4}dx + \int_{\gamma_2} \frac{1}{1+z^4}dz$

$$\begin{aligned} |\int_{\gamma_2} \frac{1}{1+z^4}dz| &\leq \text{length}(\gamma_2)\sup(\frac{1}{1+z^4}) \\ &= \pi R \sup_{\theta\in[0,\pi]\frac{1}{|1+R^4e^{4i\theta}|}} \\ &\leq \pi R(\frac{1}{e^4-1}) \to 0 \text{ as } R \to \infty \end{aligned}$$

Thus we have that: $\int_{\mathbb{R}} \frac{1}{1+x^4}dx = \frac{\pi}{\sqrt{2}}$ by the Residue Theorem.

Lecture 11 has examples for (2).

5 Zeroes and Poles

Question: Let F be a map from U open to $\mathbb{C}$. How many zeroes of f are there in U? 2 zeroes.

Theorem 5.1. (Local Argument Theorem)
Suppose either f has a pole of order k or a zero of order k at p. We also have $f : U \to \mathbb{P}^1$ holomorphic. So $p \in D(p,r) \subset D(\bar{p},r) \subset U$ such that p is the only zero or pole in $D(p,r)$, then

$$\frac{1}{2\pi i}\oint_{\partial D(p,r)} \frac{f'(\zeta)}{f(\zeta)} d\zeta = \begin{cases} k & \text{if } f \text{ has a zero of order } k \text{ at } p \\ -k & \text{if } f \text{ has a pole of order } k \text{ at } p \end{cases}$$

Proof. Well on $D(\bar{p},r)$: $f = (z-p)^k(\sum_{i=0}^{\infty} a_i(z-p)^i)$, and $f' = k(z-p)^{k-1}(\sum_{i=0}^{\infty} a_i(z-p)^i) + (z-p)^k(\sum_{i=1}^{\infty})ia_i(z-p)^{i-1}$

Then $\frac{f'}{f} = \frac{k}{z-p} + \frac{a_1}{z-p} + \dots$ and $\frac{1}{2\pi i}\oint_{\partial D(p,r)} \frac{f'(\zeta)}{f(\zeta)} d\zeta = k$. ☐

Theorem 5.2. (Global Argument Theorem)
$F : V \to \mathbb{P}^1$ holomorphic, V open, $U \subset \bar{U} \subset V$. The the number of zeroes of f in U minus the number of poles of f in U is:

$$\frac{1}{2\pi i}\int_{\partial U} \frac{f'(\zeta)}{f(\zeta)} d\zeta$$

Proof. Surround poles and zeroes with small disks D_i then $0 = \int_{\partial(U-D_i)} \frac{f'(\zeta)}{f(\zeta)} d\zeta$ *implies* $\int_{\partial U} \frac{f'(\zeta)}{f(\zeta)} d\zeta + \sum_i \int_{\partial D_i} \frac{f'(\zeta)}{f(\zeta)} d\zeta = \sum$(multiplicities) ☐

Theorem 5.3. (Open mapping theorem via an argument approach) $f : U \to \mathbb{C}$ holomorphic $\implies f(U)$ is open. That is, $\forall p \in U, \exists D(f(p),r) \subset f(U)$.

Proof. Set $g(z) = f(z) - f(p) = f(z) - q$. So p is an isolated zero of g. Then $\frac{1}{2\pi i}\int_{D(p,\epsilon)} \frac{g'(\zeta)}{g(\zeta)} d\zeta = k > 0$. $implies f^{-1}(q')$ also has k points. ☐

5.1 Rouche Theorem

Theorem 5.4. (Rouche Theorem)
Let $f, g : U \to \mathbb{C}$ holomorphic, U open, $\partial\bar{U}$ piecewise smooth, $|f(\zeta) - g(\zeta)| < |f(\zeta)| + |g(\zeta)|$. Then f and $f + g$ have the same number of zeroes in U (with multiplicity).

Proof.
(1) On $\partial\bar{U}$, $f(\zeta), g(\zeta) \neq 0$

(2) $\frac{f(\zeta)}{g(\zeta)}$ cannot lie on the negative real half axis

(3) $\forall t \in [0,t], tf(\zeta) + (1-t)g(\zeta) \neq 0$

☐

Theorem 5.5. (Herwitz Theorem)
$f_n : U \to \mathbb{C}*$ holomorphic (no zeroes on U), and $f_n \to f$ uniformly on compact sets in U. Then either $f : U \to \mathbb{C}*$ has no zeroes or $f \equiv 0$.

Proof. Suppose toward a contradiction that f has a zero but $f \not\equiv 0$. Then on some boundary of a disk D in U,

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f'}{f} = k > 0$$

But then

$$\frac{1}{2\pi i}\lim_{n\to\infty}\int_{\partial D}\frac{f_n'}{f_n}=k$$

which is a contradiction. ☐

Theorem 5.6. Let f be a rational map $f:\mathbb{P}^1\to\mathbb{P}^1$. Then $\forall p\in\mathbb{P}^1$, including ∞, we have that $\#f^{-1}(p)$ is a constant k (with multiplicity).

Proof. We separate into four cases (lecture 12):

(i) Case 1: $(p\neq\infty,\infty\notin f^{-1}(p))$

(ii) Case 2: $(p\neq\infty,\infty\in f^{-1}(p))$

(iii) Case 3: $(p=\infty,\infty\notin f^{-1}(p))$

(iv) Case 4: $(p=\infty,\infty\in f^{-1}(p))$

☐

Remark 5.1. A bijective holomorphic function has a holomorphic inverse.

Proof. We have that $f:U\to V$ is bijective and holomorphic. Then f is open and f^{-1} is continuous. Then f^{-1} is holomorphic by the open mapping theorem. ☐

5.2 General Results on Distribution

Bijective Holomorphic Maps:

1. $\mathbb{C}\to\mathbb{C}$
2. $D(0,1)\to D(0,1)$
3. $\mathbb{P}^1\to\mathbb{P}^1$

Proposition 5.1. Let $f:\mathbb{C}\to\mathbb{C}$ be bijective, holomorphic. Then $f(z)=az+b$ for $a\neq 0,b$ constants.

Proof.

Lemma 5.1. If $f:\mathbb{C}\to\mathbb{C}$ is bijective and holomorphic then $\lim\limits_{n\to\infty}|f(z)|=\infty$. That is, $\forall\epsilon>0$, $\exists c>0$ with $|z|>c\implies|f(z)|>\frac{1}{\epsilon}$.

Proof. Since f is bijective and holomorphic, its inverse f^{-1} is continuous. Then any compact set is mapped to a compact set under f^{-1}. That is, $f^{-1}(\{z:|z|<\frac{1}{\epsilon}\})$ is compact, so it is bounded and contained in $D(0,c)$ for some c. Thus, $f(\mathbb{C}\setminus D(0,c))\subset\mathbb{C}\setminus\{z:|z|<\frac{1}{\epsilon}\}$. ☐

Now we look at the function f from ∞: We will set $g(z)=\frac{1}{f(\frac{1}{z})}$. Then we have that $g:D(0,\frac{1}{c})*\to D(0,\epsilon)*$. (Recall * means that zero is excluded). Then we have that g is bounded and meromorphic at zero. We then extend it to a disk by the removable singularities theorem. Thus we have that $g:D(0,\frac{1}{c})\to D(0,\epsilon)$ and $g(0)=0$. Also, $g:D(0,\frac{1}{c})*\to D(0,\epsilon)*$ is injective. Thus $g'(0)\neq 0$ by the normal form interpretation and so $|g'(0)|=2A, A>0$. Now shrinking $D(0,\frac{1}{c})$ if necessary, we have that $|g(z)|\geq A|z|$ on $D(0,\frac{1}{c})$.

Lemma 5.2. There exists a $B, D > 0$ such that if $|z| > D$ then $|f(z)| < B|z|$.

Proof. $|f(z)| = \frac{1}{|g(\frac{1}{z})|} < \frac{1}{A|\frac{1}{z}|} = \frac{|z|}{A}$ if $|z| > c$ as desired. $\square$

This means that $|f(z)|$ has linear growth, so by Liouville's Theorem it is a linear function. $\square$

Remark 5.2. Also note that $f : \mathbb{C} \xrightarrow{\cong} \mathbb{C}$ extends to $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\infty \mapsto \infty$., and $f : D(0,1) \xrightarrow{\cong} D(0,1)$.

5.3 Maximum Modulus Principle

Lemma 5.3. (Schwartz Lemma)
$D = D(0,1).f : D \to D$ holomorphic such that $f(0) = 0$.

1. $|f'(z)| \leq 1$ and $|f(z)| \leq |z|$
2. $|f(z)| = |z|$ at a point $|f'(0)| = 1 \implies f$ is a rotation.

Proof. Set $g(z) = \frac{f(z)}{z}$, or at 0 we can either look at power series or removable singularities. Then we have that (i) is equivalent to $|g(0)| \leq 1, |g(z)| \leq 1$. And (ii) is equivalent to $|g(0)| = 1$ at a point either zero or not. On $\partial D(0, 1-\epsilon)$, $|f(z)| \leq 1$ (map sends $D(0, 1-\epsilon) \to D$). Then $|\frac{1}{z}| = \frac{1}{1-\epsilon} \implies |g(z)| \leq \frac{1}{1-\epsilon}$ on $\partial D(0, 1-\epsilon)$. But by Max Principle, the maximum is attained on $\partial D(0, 1-\epsilon)$. Thus $\forall \epsilon > 0$, $|g(z)| \leq 1$ on D (thus (i) is proven). For (ii), if $|g(z)| = 1$ at an interior point, such that g is non-constant, by the max principle, then $|g(z)| > 1$ at some point $p \in \partial D(0, 1-\epsilon)$. But this is a contradiction. Thus $|g(z)| = 1$ at a point z_0 in the interior. Then g is a rotation. $\square$

6 Special Maps

1. $Aut(\mathbb{C}) = \{z \mapsto az + b, \quad a \in \mathbb{C}^*, b \in \mathbb{C}\}$
2. $Aut(D)$
3. $Aut(\mathbb{P}^1)$

6.1 Fractional Linear Transformations

Theorem 6.1. Invertible $\mathbb{P}^1 \to \mathbb{P}^1$ maps are all of the form $z \mapsto \frac{az+b}{cz+d}$ such that the determinant is non-zero. For example, if $w = \frac{az+b}{cz+d}$ then $w^{-1} = \frac{dw-b}{-cw+a}$

All of this forms: $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\infty \mapsto a$, which implies that if $R_a(z) = \frac{1}{z-a} : a \mapsto \infty$ then $f = R_a^{-1}(bz + c)$.

6.2 Automorphisms of the Plane

Note that the $GL(2, \mathbb{C})$=invertible 2×2 matrices. We also have that $PGL(2, \mathbb{C}) = GL(2, \mathbb{C})/\mathbb{C}^*$. We have that $PGL(2, \mathbb{C}) \to Aut(\mathbb{P}^1)$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d}$. We have that the generators of $PGL(2, \mathbb{C})$ are given by:

1. $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \to (z \mapsto az)$
2. $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \to (z \mapsto z + b)$

3. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to (z \mapsto \frac{1}{z})$

Theorem 6.2. $PGL(2, \mathbb{C}) \to Aut(\mathbb{P}^1)$ is an isomorphism.

Proof. **Surjective:** We have that $f \in Aut(\mathbb{P}^1) \implies f(\infty) = c \in \infty$ or ∞.

$$\begin{cases} \text{If } f(\infty) = \infty \implies f(z) = \frac{az+b}{0z+1} \in \varphi(PGL(2, \mathbb{C})) \\ \text{If } f(\infty) = c \implies f(w) = \frac{-c(aw+b)}{-1(aw+b)+1} \in \varphi(PGL(2, \mathbb{C})) \end{cases}$$

Injective: Injective $\iff \varphi^{-1}(\mathbb{1}) =$ one point. We have that the identity in $Aut(\mathbb{P}^1)$ is given by $z \mapsto z \implies \frac{az+0}{0+a}, a \in \mathbb{C}^*$ So $\varphi^{-1}(\mathbb{1}) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ is an equivalency class of the identity matrix. □

Theorem 6.3. The images $\Im(\mathbb{P}^1)$ denoted (e, g, h) distinct of any three distinct points $(a, b, c) \in \mathbb{P}^1$ determine a unique map in $Aut(\mathbb{P}^1)$.

Proof. (i) Existence: There are many subcases here, check lecture 14

(ii) Uniqueness: A lot of subcases... lecture 14

□

Theorem 6.4. $z \mapsto \frac{az+b}{cz+d}$ maps circles and lines in $\mathbb{C}$ to circles and lines in $\mathbb{C}$.

Proof. lecture 14... lot of geometry □

6.3 The Disc

Proposition 6.1. Isomorphisms from $D(0, 1) \to D(0, 1)$:

(i) Mappings from zero to zero are all of the form $z \mapsto e^{i\theta}, \theta \in \mathbb{R}$.

(ii) The map $g(z) = \frac{z-a}{1-\bar{a}z}, a \in D(0, 1)$ maps $D(0, 1) \to D(0, 1)$ and $a \to 0$ and is invertible.

(iii) General map is given by $h(z) = e^{i\theta}(\frac{z-a}{1-\bar{a}z})$

Proof. (i) By Schwartz Lemma, for f, f^{-1} we have that $|f'(0)| = 1 \implies f(z) = e^{i\theta}z$

(ii) We consider $g(z) = \frac{z-a}{1-\bar{a}z}, a \in D \implies g(a) = 0$. Then we can compute the inverse of g as $g^{-1}(z) = \frac{z+a}{\bar{a}z+1}$.

(iii) Then the general map follows from both points.

□

6.4 The Riemann Mapping Theorem

Definition 6.1. (Simply Connected Domain) $U \subset \mathbb{C}$ is a simply connected domain if:

(i) U open

(ii) U is pathwise connected ($\forall p, q \in U$ there exists a path $f : [0, 1] \to U$ continuous, $f(0) = p$, $f(1) = q$).

(iii) $forall$ pair of paths f_1, f_2 joining p, q in U there exists $F : [0, 1] \times [0, 1] \to U$

Classification of simply connected domains up to a biholomorphic equivalence:

1. $\mathbb{C}$

2. Everything else ($\subset \mathbb{C}$) (Riemann's Theorem)

Theorem 6.5. Riemann's Mapping Theorem: Any simply connected domain in $\mathbb{C}$ that is not $\mathbb{C}$ is holomorphically that is not $\mathbb{C}$ is holomorphically equivalent to $D(0,1)$.

Proof. 1. $\mathbb{C}$ is different from $D(0,1)$. Holomorphism: $\mathbb{C} \to D(0,1)$ is constant by Liouville's Theorem.

2. Step 1: Let U be a simply connected domain not equal to $\mathbb{C}$. Want to show that there exists a $g : U \to V$ is invertible, holomorphic, and $V \subset D(0,1)$. There exists an $a \notin U$. Since U is simply connected, it implies $f(q) = \int_{p_0}^{q} \frac{1}{z-a} dz$ is well-defined (f is $log(z-a)$, and the image of f lies in a strip). For it to be well-defined, we want it to be indepedent of path. We can show this by considering the pull back.

3. Step 2: V is a simply connected domain in $D(0,1)$. We will look at $S = \{g : V \to D(0,1)\}$. Since S is not equal to the empty set, the identity maps $V \to V \in S$. Then the g we want to maximize $|g'(0)|$ for $g : V \simeq D(0,1)$. We will use Montel's Theorem: An infinite family of holomorphic functions $\{f_\alpha\} : U \to \mathbb{C}$ bounded uniformly, $|f_\alpha| < M$ on U. Then there exists a subsequence that converges $f_j \to f$ normally.

4. We will also use Ascoli Ansela Theorem: Let $f_j : K \to \mathbb{R}(\text{or } \mathbb{C})$.

   (a) f_j is uniformly bounded, $|f_j| < M$

   (b) f_j is equicontinuous

   Then there exists a subsequence converging uniformly to an $f : K \to \mathbb{R}(\text{or } \mathbb{C})$.

5. We now apply these two theorems for Riemann: Had $V \xrightarrow{i} D(0,1)$ s.t. V open subset, simply connected. We set $\mathcal{F} = \{f : V \to D(0,1) \text{ is injective}\}$. Then f bounded implies that $f(0)$ is too since $f(0) = \int_C \frac{f(\zeta)}{(\zeta-z)^2} d\zeta$. Now we take a $\hat{\mathcal{F}} \subset \mathcal{F}$ to be a maximizing sequence f_i for $|f'(0)|$.

6. Apply Montel to $\hat{\mathcal{F}}$: Now a subsequence $\tilde{f}_i$ converging to f (which is holo) with $|f'(0)| > |g'(0)| \forall g \in \mathcal{F}$. So $f(0) = 0$. For f to be injective, we want on $V \setminus \{z_0\}$ want $f(w) - f(z_0) \neq 0$. Fixing z_0: We have that $g(w) = f(w) - f(z_0)$ is the limit of $g_i(w) = f_i(w) - f_i(z_0)$. But since the f_i are injective, $\implies g : V \setminus \{z_0\} \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. But by Herwitz THeorem, we have that g is the limit of g_i so either $g : V \setminus \{z_0\} \to \mathbb{C}^*$ or $g_i \equiv 0$.

7. So since f is injective, we have that $f : V \to D(0,1)$.

8. Thus f is our candidate for $f : V \simeq D(0,1)$. Suppose not: Then we have that a point 0 is sent by f to $f(v)$. Then we create another map which sends $w \to \frac{w-R}{1-\overline{R}w}$. We also consider a function g which maps everything except that point 0. Since all of the mappings are open and simply connected, we can take a square root. We have:

$$V \xrightarrow{f} D(0,1) \xrightarrow{w \mapsto \frac{w-R}{1-\overline{R}w}} D(0,1) \xrightarrow{u \mapsto \sqrt{u}} D(0,1) \xrightarrow{v \mapsto \frac{v-\sqrt{-R}}{1-v\overline{\sqrt{-R}}}} \mapsto 0$$

   The square-root is well-defined on $\frac{f(z)-R}{1-f(z)\overline{R}}, z \in V$ because this is simply connected and avoids 0.

9. In general, $\left(\frac{w-\alpha}{1-w\overline{\alpha}}\right)' = \frac{(1-w\overline{\alpha})+\overline{\alpha}(w-\alpha)}{(1-w\overline{\alpha})^2} = \frac{1-\alpha\overline{\alpha}}{(1-w\overline{\alpha})^2}$. We will use this for $\alpha = R, \sqrt{-R}$. Derivative of f is the composition above at 0. We get that:

$$= f'(0) \times \frac{1-R\overline{R}}{1} \times \frac{1}{2\sqrt{-R}} \times \frac{1-\sqrt{-R}\overline{\sqrt{-R}}}{(1-\sqrt{-R}\overline{\sqrt{-R}})^2} = f'(0)\frac{1-|R|^2}{2\sqrt{-R}}\frac{1}{1-|R|} = f'(0)\frac{1+|R|}{2\sqrt{-R}} > 0 \text{ in norm}$$

   This is a contradiction.

∎

What are these holomorphic equivalences?

(1) D a disk or a half plane has f given by a Mobius transformation $z \mapsto \frac{az+b}{cz+d}$. So we can choose any three points $(\alpha, \beta, \gamma) \mapsto (1, i, -1)$. We need to choose these in the correct order. So we want $\frac{a\alpha+b}{c\alpha+d} = 1$, $\frac{a\beta+b}{c\beta+d} = i$, and $\frac{a\gamma+b}{c\gamma+d} = -1$.

(2) We can have strips: Taking the function e^x maps it to the upperhalf plane. Then using (1) you can map it again.

Example 6.1. Take $w = \frac{1}{2}(z + \frac{1}{z})$. It maps a circle to an airfoil (whale-like shape).

Example 6.2. Take $w = z^2$. Then the circle's angle is doubled and the length is squared, so its mapped to a cardioid. $(2a \mapsto 4a^2)$.

Example 6.3. Take $w = e^{2mi\text{arccot}(p\frac{z+1}{z-1})^{mn}}$ whose angle is $\frac{\pi}{mn}$ is mapped to the half plane.

Example 6.4. (Schwartz-Christoffel Transform) We want to map the upper half plane points $x_1, x_2, \ldots$ to a polygon with angles $\alpha_1, \alpha_2, \ldots$ and vertices $w_1, w_2, \ldots$. This is given by $\frac{dw}{dz} = A(z - x_1)^{\frac{\alpha_1}{\pi}-1}(z - x_2)^{\frac{\alpha_2}{\pi}-1} \ldots (z - x_n)^{\frac{\alpha_n}{\pi}-1}$.

7 Extending A Map

7.1 Analytic Continuation

7.2 Riemann Surfaces

Instead of living in $U \subset \mathbb{C}$, we will consider a general Riemann surface.

Definition 7.1. (Riemann Surface) A Riemann surface is defined as open bits of $\mathbb{C}$ that are glued together with biholomorphic glue. More rigorously, it is a connected one-dimensional manifold. You need to be careful with the "glue" because you want to avoid double-points (it will not create a Hausdorff space). Its better to start with a pre-existing surface and put points on it. It is a surface covered by coordinate charts $(U_i \subset S, f_i : U_i \to V_i \subset \mathbb{C})$ such that the f_i and f_j are holomorphically compatible.

Example 7.1. One natural venue: the locals for multiply defined functions. Like $\sqrt{z}$. So $\sqrt{z}$ wind once around zero changes signs. Instead, we can look at $w^2 = z$ in $\mathbb{C}^2$. This surface for the $w^2 = z$ is well-defined, because it is just the coordinate w on a curve S.

Remark 7.1. "Because of budget cuts at McGill there are no four-dimensional blackboards" -Prof. Hurtubise.

Example 7.2. (Example of Compact Riemann Surfaces) Any compact surface in $\mathbb{R}^3$ (they are classified by genus (which is the number of holes)). We have that $S \to \mathbb{R}^3$ inherits a metric by restriction. $T_pS^2 \to T_p\mathbb{R}^3 \simeq \mathbb{R}^3$.

Theorem 7.1. There exist isothermal coordinates on S locally. That is, $g = a(x, y)(dx^2 + dy^2)$ such that $z = x + iy$.

Definition 7.2. The genus is the number of holes in the Riemann surface.

$g = 1$ curves are obtained by quotienting $\mathbb{C}$ by a lattice. Two points are said to be equivalent if they are translates under this lattice.

(1) Some built as a domain of definition of a holomorphic function.

Example 7.3. $\sqrt{z}$ It passes through itself while wrapping around. It is the curve $S : w^2 = z \in \mathbb{C}^2$.

Example 7.4. $\log x$ As the curve $S : e^w = z \in \mathbb{C}^2$

(2) Compact orientable surface $\subset \mathbb{R}^3 \implies$ it has a metric on S, and it admits a scalar product $< \cdot, \cdot >$ on the Tangent plane $T_pS \implies$ that is has a conformal structure.

Definition 7.3. (Conformal structure) Metrics are not scalar. That is, $< \cdot, \cdot > \simeq e^{f(x,y)} < \cdot, \cdot >$.

(3) A "curve" in $\mathbb{C}^2$ at out by a holomorphic equation. $g(x, y) = 0$.
(4) We observe two cases:

1. genus zero, compact: Riemann sphere $\mathbb{P}^1(\mathbb{C})$, it is unique
2. genus one: elliptic curves, build as $\mathbb{C}$ mod (2 independent translations). That is, $\mathbb{C}/\tau_1\mathbb{Z} + \tau_2\mathbb{Z}$ by viewing $\mathbb{C}$ as a group.

8 A Quick Tour of Elliptic Functions

Definition 8.1. (Fundamental domain Δ) A fundamental domain Δ for $\mathbb{C}/\tau_1\mathbb{Z} + \tau_2\mathbb{Z}$ is given by

$$\{a\tau_1 + b\tau_2 : a, b \in [0, 1]\}$$

$\mathbb{C}/\tau_1\mathbb{Z} + \tau_2\mathbb{Z} = \Delta/\sim$ given by $(0\tau_1 + b\tau_2) \sim (1\tau_1 + b\tau_2)$ and $(a\tau_1 + 0\tau_2) \sim (a\tau_1 + 1tau_2)$. Any translate of Δ would work too. Moreover, can change the basis by $\mathbb{Z}$ coefficients.

Definition 8.2. (Elliptic functions) They are on $\mathbb{C}/\tau_1\mathbb{Z} + \tau_2\mathbb{Z}$ and are normalized to $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$.

Proposition 8.1. There is an equivalence between:

1. Doubly periodic { Holomorphic or Meromorphic} functions on $\mathbb{C}$, periods $\tau_1\mathbb{Z} + \tau_2\mathbb{Z}$.
2. { Holomorphic or meromorphic} functions on $\mathbb{C}/\tau_1\mathbb{Z} + \tau_2\mathbb{Z}$ (elliptic functions)
3. { holomorphic or meromorphic } functions on Δ with boundary $f(b\tau_2) = f(\tau_1 + b\tau_2), b \in [0, 1]$ and $f(a\tau_1) = f(a\tau_2 + \tau_2), a \in [0, 1]$.

Indeed, holomorphic function f on $\mathbb{C}/\tau_1\mathbb{Z} + \tau_2\mathbb{Z} \equiv$ holomorphic function on Δ and the boundary conditions. But Δ is compact, so f bounded on Δ means the periodicity of f is bounded on $\mathbb{C}$, so f is constant by Liouville's.

Functions with poles: Suppose f has poles at $p_i \in \mathring{\Delta}$ (interior). Compute:

$$\frac{1}{2\pi i} \int_{\partial\Delta} f = \frac{1}{2\pi i} \int_A^B f dz - \int_B^C f dz + \int_C^D f dz + \int_A^D f dz = 0$$

Proposition 8.2. (No poles on $\partial\Delta$) $\sum_{p_i \in \Delta} \mathrm{Res}_{p_i}(f) = 0$

Corollary 8.1. If f has at most one pole of multiplicity 1, then f is constant.

Now look at $\frac{1}{2\pi i} \int_{\partial\Delta} \frac{f'}{f} dz$. This gives you the number of zeroes minus the number of poles in Δ. This is equal to:

$$\frac{1}{2\pi i} \int_{\partial\Delta} \frac{f'}{f} dz = 0$$

Proposition 8.3. f has the same number of zeroes as poles. So f is a meromorphic function on $\mathbb{C}/\tau_1\mathbb{Z} + \tau_2\mathbb{Z}$.

Next consider

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial\Delta}\frac{zf'(z)}{f(z)}dz &= \frac{1}{2\pi i}\left(\int_A^B \frac{zf'(z)}{f(z)}dz + \int_B^C \frac{zf'(z)}{f(z)}dz + \int_C^D \frac{zf'(z)}{f(z)}dz + \int_D^A \frac{zf'(z)}{f(z)}dz\right) \\
&= \frac{1}{2\pi i}\left(\int_A^B \frac{zf'(z)}{f(z)} - \frac{(z+\tau_2)f'(z)}{f(z)}dz + \int_C^D \frac{zf'(z)}{f(z)} - \frac{(z-\tau_1)f'(z)}{f(z)}dz\right) \\
&= \frac{1}{2\pi i}\left(-\tau_1\int_A^B \frac{f'(z)}{f(z)}dz + \tau_1\int_C^D \frac{f'(z)}{d(z)}\right) \\
&= \frac{1}{2\pi i}\left(-\tau_2[log(f(z))]_A^B + \tau_2[log(f(z))]_C^D\right)
\end{aligned}$$

Thus, f being the same at $A, B, C, D \implies log(f)$ same (mod $2\pi i\mathbb{Z}$) at $A, B, C, D \implies$

$$\frac{1}{2\pi i}\int_{\partial\Delta}\frac{zf'}{f}dz = n_2\tau_2 + n_1\tau_2$$

Theorem 8.1. (Abel's Theorem) If f has zeroes at $z_1, \ldots, z_n$ with multiplicity so that a double zero at z_1 gives z_1, z_1. And poles $p_1, \ldots, p_n$ so that multiplicity so that a double pole at p_1 gives p_1, p_1. then $\sum z_i - \sum p_i = n_2\tau_2 + n_1\tau_1$. That is, $\sum z_i - \sum p_i = 0$ in $\mathbb{C}/\tau_1\mathbb{Z} + \tau_2\mathbb{Z}$.

Are there any meromorphic functions at all?

1. Can normalize (τ_1, τ_2) by sending $\tau_1 \mapsto 1$.
2. Send $z \mapsto z/\tau_1$ so we are left with $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$

Theorem 8.2. (Weierstrass $\mathfrak{P}$-function) Try for a double pole at 0 (no residue).
We will try $\sum\limits_{m,n\in\mathbb{Z}} \frac{1}{(z+m+n\tau)^2}$ but this does NOT work. We observe layers of $\mathbb{Z} + \tau\mathbb{Z}$ on the integer lattice (each layer scales by 8 points..., so at level k there are 8k points). We try to enscribe and circumscribe a sphere on this lattice, the inner sphere only contains the origin, while the outer contains all points of layer 1. So $p \in$ level $k \implies k_1 k \geq ||p|| < k_2 k$ where k_1 is the radius of the inner sphere and k_2 is the radius of the outersphere. We tried:

$$\sum_{m,n\in\mathbb{Z}} \frac{1}{(z-m-n\tau)^2}$$

fails because it does not converge. Instead we will try:

$$\frac{1}{z^2} + \sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(z-m-n\tau)^2} - \frac{1}{(m+n\tau)^2} = \frac{1}{z^2} + \sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)} \frac{-z^2+2z(m+n\tau)}{(z-m-n\tau)^2-(m+n\tau)^2}$$

Let $|z| < C$ and $p \in k-$th layer $\implies |z-p| > k_1 k - c$ and if $k >> 0 \implies k_1 k > k_1' k, k_1' > 0$

$$\begin{aligned}
&\implies \frac{1}{|z-m-n\tau|} < \frac{1}{k'k} \quad \frac{1}{|m+n\tau} < \frac{1}{k'k} \\
&\implies |-z^2+2z(m+n\tau)| < c'|m+n\tau| < c'k_2 k
\end{aligned}$$

So we have that

$$\begin{aligned}
\frac{1}{z^2} + \sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)} \frac{-z^2+2z(m+n\tau)}{(z-m-n\tau)^2-(m+n\tau)^2} &\leq \frac{1}{|z|^2} + \sum_{\text{levels}}\left((\text{number of points in level}) \times \frac{1}{(k'k)^4}c'k_2 k\right) \\
&= \sum \frac{1}{k^2} \text{ converges as desired}
\end{aligned}$$

Properties of these functions:

1. $\mathfrak{p}$-functions are even ($\mathfrak{p}(z) - \mathfrak{p}(-z)$)
2. because they are even we have that $\frac{1}{(z-m-n\tau)^2} + \frac{1}{(z+m+n\tau)^2}$ these two terms have the same value for z and $-z$
3. they are also periodic

Proposition 8.4. The $\mathfrak{p}$-function is periodic.

Proof.

$$\mathfrak{p}'(z) = \sum_{m,n} \frac{-2}{(z-m-n\tau)^3}$$

So it is periodic with periods $1, \tau$. But then

$$\mathfrak{p}(z+1) - \mathfrak{p}(z) = \int_z^{z+1} \mathfrak{p}'(\zeta) d\zeta$$

and on the half-lattice, they have the same pole, so they cancel out. That is,

$$\mathfrak{p}(z) = \frac{1}{z^2} + \sum \frac{1}{(z-m-n\tau)^2} - \frac{1}{m+n\tau}$$

$$\mathfrak{p}(z+1) = \frac{1}{(z+1)^2} + \sum \frac{1}{(z-m+1-n\tau)^2} - \frac{1}{m+n\tau}$$

So $\mathfrak{p}(z+1) - \mathfrak{p}(z)$ is holomorphic on a compact set, so it is also bounded. But then

$$\begin{aligned}\mathfrak{p}(z+m+n\tau+1) - \mathfrak{p}(z+m+n\tau) &= \int_{z+m+n\tau}^{z+m+1+n\tau} \mathfrak{p}'(\zeta) d\zeta \\ &= \int_z^{z+1} \mathfrak{p}'(\zeta) d\zeta\end{aligned}$$

But $\mathfrak{p}(z+1) - \mathfrak{p}(z)$ bounded, periodic implies constant by Lioville's Theorem. So now consider $\mathfrak{p}(1/2) - \mathfrak{p}(-1/2) = 0$ because $\mathfrak{p}$ is even. This means that $\mathfrak{p}(z+1) = \mathfrak{p}$, so it is periodic. The same proof can be used for $\mathfrak{p}(z+\tau) = \mathfrak{p}(z) \quad \forall z$ □

Now expand the function into its Laurent Series:

$$\mathfrak{p}(z) = \frac{1}{z^2} + a_0 + a_2 z^2 + a_4 z^4 + \dots$$

We also have that:

$$f(z) = \mathfrak{p}(z) - \frac{1}{z^2} = \sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(z-m-n\tau)^2} + \frac{1}{(z+m+n\tau)}$$

But then this means that:

$$f(0) = \sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(z-m-n\tau)^2} + \frac{1}{(z+m+n\tau)} = 0$$

We have that:

$$\mathfrak{p}'(z) = \frac{-2}{z^3} + 2a_2 z + 4a_4 z^3 + \dots$$

$$(\mathfrak{p}'(z))^2 = \frac{4}{z^6} - \frac{8a_2}{z^2} - 16a_4 + o(z^2)$$

Now consider the cube of the function:

$$(\mathfrak{p}(z))^3 = \frac{1}{z^6} + \frac{3a_2}{z^2} + 3a_4 + o(z^2)$$

Now consider:

$$(\mathfrak{p}'(z))^2 - 4(\mathfrak{p}(z))^3 = \frac{-20a_2}{z^2} - 28a_4 + o(z^2)$$

Take:

$$(\mathfrak{p}'(z))^2 - 4(\mathfrak{p}(z))^3 + 20a_2\mathfrak{p}(z) + 28a_4 = o(z^2)$$

But then since the left hand side is double periodic, so is $o(z^2)$. Thus:

$$(\mathfrak{p}'(z))^2 - 4(\mathfrak{p}(z))^3 + 20a_2\mathfrak{p}(z) + 28a_4 = o(z^2) = 0$$

Set $x = \mathfrak{p}(z)$ and $y = \mathfrak{p}'(z)$. Then $z \mapsto (x, y)$ maps $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ to the curve over complex numbers: $y^2 = 4x^3 - 20a_2x - 28a_4$ in $\mathbb{C}^2$. Then $y = \sqrt{4x^3 - 20a_2x - 28a_4}$. So:

$$\int_{x=c}^{c} \frac{dx}{dy} = \int_{c}^{c'} \frac{dx}{\sqrt{4x^3 - 20a_2x - 28a_4}}$$

by change of variables $(x = \mathfrak{p}(z), dx = \mathfrak{p}'(z)dz)$

$$= \int \frac{\mathfrak{p}'(z)}{\mathfrak{p}'(z)} dz = \int dz$$

Theorem 8.3. The map $\varphi : (\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \setminus 0) \to \mathbb{C}^2$ such that $z \mapsto (\mathfrak{p}(z), \mathfrak{p}'(z))$

(1) has image in $\Sigma \subset \mathbb{C}^2$ such that $\Sigma : y^2 = 4x^3 - 20a_2x - 28a_4$

(2) maps $0 \mapsto \infty$

(3) and the map is bijective

Proof. (1) Did previously

(2) 0 is the only pole

(3) Consider $S = (\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \setminus 0)$. Then we take $S \to \Sigma \leftrightarrow \mathbb{C}^2 \xrightarrow{(x,y)\mapsto(x)} \mathbb{C}$ We also have $S \xrightarrow{\mathfrak{z}\mapsto x} \mathbb{C}$. Look at $(\mathfrak{p}(z) - c)$ has double pole, so it has two genuses z_1, z_2. The locations of the poles sum to zero, which implies that $z_1 = -z_2$. Thus $\mathfrak{p}$-function is surjective onto $\mathbb{C}$ and $\mathfrak{p}^{-1}(c) = \{z_1, -z_1\}$. But, $\mathfrak{p}'(z)$ is an odd function. So $\mathfrak{p}'(z_1) = -\mathfrak{p}'(-z_1)$. So these are two values unless $\mathfrak{p}'(z_1) = -\mathfrak{p}'(-z_1) = 0$. We are in $y^2 = 4x^3 - 20a_2x - 28a_4$, which has a symmetry: $y \mapsto -y$ and $x \mapsto -x$.
At $\mathfrak{p}' \neq 0$: there are two pre-images of c in $\mathbb{C}$ corresponding to $-z_1, z_1$. Moreover, these are the only two points going to $\mathbb{C}$.
At $\mathfrak{p}' = 0$: there is a single point; these are the points where $z_1 = -z_1 \bmod(\mathbb{Z} + \tau\mathbb{Z})$ (these are half periods).

□

Proposition 8.5. $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ is a group.

Proof. 1. Inverse: $z \mapsto -z$

2. Neutral element: 0

3. Addition in $\mathbb{C}$: $(a,b) \mapsto a+b$.

equivalent to giving for any a, b the c with $a+b+c=0$ once you have the inverse (that is, $a+b = -c$). $\square$

In $y^2 = 4x^3 - 20a_2x - 28a_4$ we have $(x,y) \mapsto (x,-y)$ which corresponds to $z \mapsto -z \in \Sigma$. Now we look at $l \cap S = \{y^2 = 4x^3 - 20a_2x - 28a_4\}$. The intersection is given by $a\mathfrak{p}(z) + b\mathfrak{p}'(z) + c = 0$. This is doubly periodic, poles only at $z=0$, which means that the zeroes of $a\mathfrak{p}(z) + b\mathfrak{p}'(z) + c = 0$ sum to zero (in $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$).

- We have on $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ constants, $\mathfrak{p}(z)$, and $\mathfrak{p}'(z)$
- How about getting all meromorphic functions (remember $\sum_{p_i} \text{Res}_{p_i}(f) = 0$)
- Principle part of f at z_0 is the negative part of its Laurent series at z_0 (f meromorphic implies negative part is finite)
- A meromorphic function on $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ is determined by its principle parts up to constants

  Proof. If the principle part of f_1 is equal to the principle part of f_2, it implies that $f_1 - f_2$ is holomorphic on $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, which is a constant by Liouville's. $\square$

Principle Parts at 0:

Principle Parts	Function
$\frac{1}{z^2}$	$p(z)$
$\frac{1}{z^3}$	$-\frac{1}{2}p'(z)$
$\frac{1}{z^4}$	$\frac{1}{6}p''(z)$

Principle Parts at C:

Principle Parts	Function
$\frac{1}{(z-c)^2}$	$p(z-c)$
$\frac{1}{(z-c)^3}$	$-\frac{1}{2}p'(z-c)$
$\frac{1}{(z-c)^4}$	$\frac{1}{6}p''(z-c)$

Taking linear combinations we get all positive configuartions of principle parts with no simple pole terms.

Simple Poles: Weierstrass ζ-functions are functions on $\mathbb{C}$ with $\zeta' = \mathfrak{p}$ and ζ odd.

First try $\hat{\zeta_{z_0}}(z) = \int_{z_0}^{z} \mathfrak{p}(z)dz$. These are independent of path because $\mathfrak{p}(z)$ has no residues.

But then $\hat{\zeta_{z_0}}(z+1) - \zeta_{z_0}(z) = -\eta_1$ which is some constant. We have that η_1 is independent of z.

Note that it can be shown that $\hat{\zeta_{z_0}}(z+\tau) + \hat{\zeta_{z_0}}(z) = -\eta_2$ for similar reasons.

But then $\hat{\zeta_{z_0}}(z) = \hat{\zeta_{-z_0}}(z) + \int_{z_0}^{-z_0} p(\Psi)d\Psi$.

So then $\mathfrak{p}$ is even implies that $\hat{\zeta_{z_0}}(z) = -\hat{\zeta_{-z_0}}(-z) = -\hat{\zeta_{z_0}}(-z) - \int_{z_0}^{-z_0} \Psi d\Psi$.

So then we have that $\hat{\zeta_{z_0}}(z) - \frac{1}{2}\int_{z_0}^{-z_0} \Psi d\Psi = -(\hat{\zeta_{z_0}}(-z) - \frac{1}{2}\int_{z_0}^{-z_0} \Psi d\Psi)$

So if we expand: $\hat{\zeta_{z_0}}(z) - \frac{1}{2}\int_{z_0}^{-z_0} \Psi d\Psi = -\frac{1}{2} + \ldots + bz^3cz^5 + \ldots$.

Now at $\zeta(z) = -\hat{\zeta_{z_0}}(z) + \frac{1}{2}\int_{z_0}^{-z_0} \Psi d\Psi$ we have that $\zeta(z+1) - \zeta(z) = \eta_1$ and $\zeta(z+\tau) - \zeta(z) = \eta_2$. We have $\zeta(z)$ odd, simple pole at 0, Res=1.

But now take a_i constants, $\sum_{i=1}^{N} a_i = 0$ with $\mathfrak{p}_i \in \mathbb{C}$ representing $\tilde{\mathfrak{p}_i} \in \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$.

Consider $\sum a_i \zeta(z - \mathfrak{p}_i)$. We have that:

$$\sum a_i \zeta(z - p_i + 1) - \sum a_i \zeta(z - p_i) = \sum a_i \cdot \eta_1 = 0$$

And we have that:
$$\sum a_i\zeta(z-p_i+\tau)-\sum a_i\zeta(z-p_i)=\sum a_i\cdot\eta_2=0$$

This means that $f(z)$ is periodic, it has simple poles at p_i, residues a_i.

Sums and Products: So for $\sum a_m z^m$ sums of functions $a_m z^m$. We can also take $f_i(z)$, consider:
$$\sum_{i=1}^{\infty} f_i(z)$$

But can also try
$$\prod_{i=1}^{\infty} f_i(z)$$

- Suppose we want a polynomial with zeroes at a_1, a_m.
- We take the product $\prod_{i=1}^{m}(z-a_i)$
- But if this is infinity not m, it can be difficult to compute, which motivates our next topic

So we look at: $a\cdot a^2\cdot a^3\dots$. If $a=1$ then product is 1. If $|a|<1$ then product tends to 0. If $|a|>1$ product tends to infinity.

Look at the product
$$\prod_{i=1}^{N}\exp\{a_i\}=\exp\{\sum_{i=1}^{N}a_i\}$$

So if $\lim(\sum a_i)$ makes sense, $\lim\prod\exp\{a_i\}$ makes sense too.

If $\sum^{\infty} a_i$ exists then $a_i\to 0$ so $\exp\{a_i\}\to 1$.

Lemma 8.1. If $x\in\mathbb{R}$, $0<x<1$, then $1+x<e^x<1+2x$.

Proof. $[e^x-(1+x)]'=e^x-1>0$, for $x>0$. So $e^0-(1+0)=0 \implies (e^x-(1+x))>0$ for $x>0$. e^x convex and $[e^x]''>0$ implies that $1+2x>e^x$. □

So we have that for $|a_i|<1$,
$$\exp\{\frac{1}{2}\sum^{m}|a_i|\}\le\prod^{m}(1+|a_i|)\le\exp\{\sum^{m}|a_i|\}$$

So on each term:
$$\exp(\frac{|a_i|}{2})\le 1+\frac{2|a_i|}{2}$$

This implies the following proposition:

Proposition 8.6. $\sum\limits_{i=1}^{\infty}|a_i|<\infty$ converges $\iff$ $\prod\limits_{i=1}^{\infty}(1+|a_i|)$ converges.

What about $\prod\limits_{i=1}^{\infty}(1+a_i)$ for $a_i\in C$?

Problem: Some of the a_i terms might be -1, which kills the product.

So you could have $\prod\limits_{i=1}^{\infty}(1+a_i)=0$ but $\prod\limits_{i=1}^{n}(1+a_i)$ not being Cauchy.

Definition 8.3. (Notions of Convergence) $\prod\limits_{j=1}^{\infty}(1+a_j)$ converges $\iff$

(1) Only a finite number of a_i are -1, say $a_{i_1}, \ldots, a_{i_k}$ for $i_1 < i_2 < \ldots < i_k$

(2) For $N_0 > i_1, \ldots, i_k$, $\lim_{n\to\infty} \prod_{j=N_0+1}^{N} (1 + a_j)$ exists

(3) This limit is non-zero

then set $\prod_{j=1}^{\infty}(1 + a_j) = \prod_{j=1}^{N_0}(1 + a_j) \cdot \lim_{n\to\infty} \prod_{j=N_0+1}^{n} (1 + a_j)$

Lemma 8.2. If $P_N = \prod_{j=1}^{N}(1 + a_j)$ and $\tilde{P_N} = \prod_{j=1}^{N}(1 + |a_j|)$. Then $|P_N - 1| \leq \tilde{P_N} - 1$.

Proof.

$$P_N = 1 + \sum_{\text{subsets } \{i_1 \ldots i_l\} \text{ of } \{1,\ldots,n\}} a_{i_1} \ldots a_{ij}$$

$$\tilde{P_N} = 1 + \sum_{\text{subsets}} |a_{i_1}| \ldots |a_{i_j}|$$

But $\sum |a_{i_1}, \ldots, a_{i_j}| \leq \sum |a_{i_1}| \ldots |a_{i_j}|$. Thus our desired result. □

Proposition 8.7. Absolute convergence $\Longrightarrow$ convergence. That is, $\prod_{j=1}^{\infty}(1+|a_j|)$ converges $\Longrightarrow$ $\prod_{j=1}^{\infty}(1+a_j)$ converges.

Proof. Want

(i) Only a finite number of $a_j = -1$ but then $\prod(1+|a_j|)$ converges $\Longleftrightarrow$ $\sum|a_j|$ converges $\Longrightarrow$ $|a_j| \to 0$.

(ii) Have $a_j \neq -1$ if $j > N_0$. Set $Q_j = \prod_{N_0+1}^{J}(1 + a_j)$ and $\tilde{Q_j} = \prod_{N_0+1}^{j}(1 + |a_j|)$. We will look at for $M > N$, $Q_M - Q_N = Q_N(\prod_{N+1}^{M}(1 + a_j) - 1)$. So in absolute value:

$$|Q_M - Q_N| = |Q_N|| \prod_{m+1}^{\infty}(1+a_j)-1| \leq |Q_N|| \prod_{m+1}^{\infty}(1+|a_j|)-1| \leq |Q_N|| \prod_{m+1}^{\infty}(1+|a_j|)| = |\tilde{Q_M} - \tilde{Q_N}|^{-1}$$

(iii) Last thing to check is that

$$\lim_{N\to\infty} \prod_{j=1}^{N}(1 + a_j) \leq 0$$

Since $\tilde{Q_M}$ converges, this implies that

$$|\tilde{Q_M}/\tilde{Q_N} - 1| = \prod_{j=N}^{M}(1 + |a_j|) - 1 \xrightarrow{N\to\infty} 0$$

This implies that

$$\prod_{N}^{M}(1 + |a_j|) - 1 < \frac{1}{2}$$

By our lemma, we have:

$$|\prod_{N}^{M}(1 + a_j)| - 1 < \prod_{N}^{M}(1 + |a_j|) - 1 < \frac{1}{2}$$

□

Corollary 8.2. $\prod_{j=1}^{\infty}(1 + a_j)$ converges if $\sum |a_j|$ converges.

Proof. Follows from the previous proposition and lemma. □

8.1 Products of Functions

Recall that: if $f : U \to \mathbb{C}$ holomorphic, vanishes at $a_{11}, \ldots, a_{jJ}$ with an accumulation point in U then $f \equiv 0$.

So for $\prod(1 + f_i(z))$, only allow a finite number of f_i, z_i in compact subsets $K \subset U$ with $f_i(z_i) = -1$.

Theorem 8.4. f_j is a sequence of holomorphic functions on U open subset of $\mathbb{C}$. If $\sum\limits_{j=1}^{\infty} |f_j|$ converges uniformly on compact sets $K \subset U$ then $\prod_{j=1}^{\infty}(1 + f_j(z))$ converges uniformly on $K \subset U$.

Proof. (i) Given $K \subset U$ compact. We have that $\sum_{j=N}^{\infty} |f_j| < \frac{1}{2}$ for N large uniformly $\implies$ only a finite number of $f_j(z_j) = -1$.

(ii) Set $F_M = \prod_{j=1}^{M}(1 + f_j(z))$. Then if $N > M$ consider:

$$\begin{aligned} F_M - F_N &= \prod_{j=1}^{N}(1 + f_j(z))(\prod_{N}^{M}(1 + f_j(z)) - 1) \\ &\leq |F_N||(\exp(2\sum_{N}^{M} |f_j(z)|)) - 1| \end{aligned}$$

So if N large enough, $|f_j(z)| < 1$, recall we had $1 + x < \exp(2x)$ if $x \in [0, 1]$.

□

Given $z_1, \ldots, z_n$ we have that $\prod_{i=1}^{n}(z - z_i)$ vanishes at $z_1, \ldots, z_n$. What about being given an infinite sequences $z_1, \ldots, z_n, \ldots$.

Definition 8.4. (Weierstrass Functions) The trick: You have $(1 - z)\exp(-\log(1 - z))$ This is bad at $z = 1$ but it is equal to $\frac{1-z}{1-z} = 1$ on $D(0, 1)$. Then this is:

$$(1 - z)\exp(z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots + \frac{z^p}{p} + \ldots)$$

Set $E_p(z) = (1 - z)(\exp(z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots + \frac{z^p}{p}))$ truncated. It is defined on all of $\mathbb{C}$. It has a unique zero at $z = 1$. And it is "close" to 1 on $D(0, 1)$.

Lemma 8.3. For $|z| < 1$, $|1 - E_p(z)| \leq |z^{p+1}|$.

Proof. Set $E_p(z) = 1 + \sum\limits_{n=1}^{\infty} b_n z^n$. Claim: $b_n = 0$ for $1 \leq n \leq p$ and $b_n \leq 0$ for $n > p$. Calculate

$$E'_p(z) = -\exp(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p}) + (1 - z)(1 + z + z^2 + \ldots + z^{p-1})\exp(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p})$$

Note that $(1 - z)(1 + z + z^2 + \ldots + z^{p-1}) = 1 - z^p$. So then we get that:

$$E'_p(z) = -z^p \exp(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p})$$

See

(i) Taylor for $E'_p(z)$ starts at z^p

(ii) Higher order Taylor coefficients are all negative

But then $E_p'(z) = \sum np_n z^{n-1}$. This implies that the coefficients of $E_p(z)$ are such that $b_n = 0$ for $n \leq p$ and $b_n < 0$ for $n > p$. Our next step: We have $0 = E_p(1) = 1 + \sum\limits_{n=p+1}^{\infty} b_n$. But these are all negative, so this implies that $\sum\limits_{n=p+1}^{\infty} b_n = -1, b_n < 0 \quad \forall n > p \implies \sum_{n=1}^{\infty} |b_n| = 1$. This means that

$$|E_p(z) - 1| = |\sum_{n=p+1}^{\infty} b_n z^n| = |z^{p+1}|| \sum_{n=p+1}^{\infty} b_n z^{n-p-1}|$$

Now if $|z| < 1$, we have that:

$$|E_p(z) - 1| \leq |z^{p+1}| \sum_{n=p+1}^{\infty} |b_n| = |z^{p+1}|$$

□

Theorem 8.5. Suppoes sequence a_i such that $a_i \neq 0$ and a sequence $p_i \in \mathbb{N}$. Then $|a_i| \to \infty$ "fast enough" means:

- No finite accumulation point
- $\forall r > 0, \sum\limits_{j=1}^{\infty} (\frac{r}{|a_j|})^{p_j+1} < \infty$ (finite)

Then $\prod\limits_{n=1}^{\infty} E_{p_m}(\frac{z}{a_n})$ converges uniformly on compact sets to F entire. Zeroes of F are the a_i counted with multiplicity.

Proof. Fix r. We have that $\exists m_0 |a_m| > r$ for $m \geq n$. But then for $n > m_0$, $|E_{p_n}(\frac{z}{a_n}) - 1| \leq |\frac{z}{a_n}|^{p_n+1}$ for $\frac{z}{a_n} < 1$. That is, $|z| < a_m$. This implies that it holds for $z \in D(0, r)$. But then, $|\frac{z}{a_n}|^{p_n+1} < |\frac{r}{a_n}|^{p_n+1}$. This implies that: $\sum_{n=1}^{\infty} |E_{p_n}(\frac{z}{a_n}) - 1|$ converges uniformly on $D(0, r)$. This implies that $\prod_{n=1}^{\infty} (E_{p_n}(\frac{z}{a_n})) - 1 + 1$ converges uniformly. □

Corollary 8.3. Let $a_n \in \mathbb{C}$ with no accumulation point, $a_n \neq 0$. Then $\exists f$ an entire function with a zero set precisely equal to $\{a_n\}_{n=1}^{\infty}$ (always counting with multiplicity).

Proof. Fix r. Then $\exists N$ such that $\forall n > N$, $|a_n| > 2r$. Then $\sum_{m=N}^{\infty} (\frac{r}{|a_m|})^n \leq \sum_{m=N}^{\infty} (\frac{1}{2})^n < \infty$ Thus we can apply the theorem with $p_n = n - 1$. □

Remark 8.1. We can add zeroes at $z = 0$ by setting $\hat{f} = z^n f$.

Recall facts about Entire Functions:

1. We saw that we could build entire functions with arbitrary zeroes a_n counted with multiplicity, where a_n has no accumulation point in $\mathbb{C}$.
2. If $\{a_n\}$ is a finite set: we can use the polynomial $\prod(z - a_n)$
3. If $\{a_n\}$ is an infinite sequence: If $a_n \neq 0$ then $\prod\limits_{n=1}^{\infty} E_{n-1}(\frac{z}{a_n})$ entire and precisely at a_n.
4. For extra zeroes at $z = 0$, take $z^m(\prod_{n=1}^{\infty} E_{n-1}(\frac{z}{a_n}))$

Remark 8.2. If you want zeroes at a_n, poles at b_n (no accumulation point in $\mathbb{C}$), you can take:

$$\frac{\prod_{n=1}^{\infty} E_{n-1}(\frac{z}{a_n})}{\prod_{n=1}^{\infty} E_{n-1}(\frac{z}{b_n})}$$

Theorem 8.6. (Weierstrass Approximation Theorem) Suppose f is entire, with zeroes at a_n such that the zeroes are of order m at 0, [implying that a_n has no accumulation point, else $f \equiv 0$]. Then we can write:

$$f(z) = z^m E_{n-1}(\frac{z}{a_n})e^{g(z)}$$

Proof. $\frac{f(z)}{h(z)}$ has no zeroes meaning that it has a log. We take:

$$g(z) = \log \frac{f(z_0)}{h(z_0)} + \int_{z_0}^{z} \frac{[f(\zeta)/h(\zeta)]'}{f(\zeta)/h(\zeta)} d\zeta \implies e^{g(z)} = \frac{f(z_0)}{h(z_0)} + \frac{f(z)/h(z)}{f(z_0)/h(z_0)}$$

□

What about $f : U \to \mathbb{C}$?

1. If it has a finite set of zeroes, it is a polynomial
2. If it has an infinite sequence of zeroes (with multiplicity), then $a_i \in U, a_i \neq 0$

Theorem 8.7. Let $a_i \in U$ have no accumulation in U. Then there exists $f : U \to \mathbb{C}$ whose zeroes are precisely the a_i.

Proof. We first being by considering the set U such that $p \neq a_i \quad \forall i$, and we map $z \mapsto \frac{1}{z-p}$ to a set $\hat{U}$ within $\mathbb{P}^1$. Then we have three cases:

1. $\infty \in \hat{U} \subset \mathbb{P}^1$
2. $\mathbb{P}^1 - \hat{U}$ is compact, infinite piece of $\mathbb{C}$
3. $a_j \in \hat{U}$, $a_j \neq \infty$ with new a_j's (could write $\hat{a_j}$ but too lazy)

Each a_j has a nearest point $\hat{a_j}$ in $\mathbb{P}^1 \setminus \hat{U}$, with ($d(a_j, z)$ continuous on $\mathbb{P}^1 \setminus \hat{U}$ compact, minimum attained) such that $d(a_j, \hat{a_j}) = d_j$ and $d_j \to 0$. Choose K compact subset of $\hat{U}$. But then we have that $d(K, \mathbb{P}^1 \setminus \hat{U}) = \delta > 0$ because $K \times (\mathbb{P}^1 \setminus \hat{U})$ is compact, and $d(x, y)$ is continuous there. This implies that $|z - \hat{a_j}| > \delta$ for $z \in K$ and $d_j \to 0 \implies \delta \geq 2d_{i_j}$ for some $i > i_0$. Thus we have that:

$$\frac{|a_j - \hat{a_j}|}{|z - \hat{a_j}|} < \frac{1}{2}$$

Now we can look at $\hat{f(z)} = \prod_{j=1}^{\infty} E_j(\frac{a_j - \hat{a_j}}{z - \hat{a_j}})$. Since E_j is zero at 1, we have that $|E_j(\omega)| < |\omega|^{j+1}$ for $\omega < 1$. This implies that:

$$\sum | - 1 + E_j(\frac{a_j - \hat{a_j}}{z - \hat{a_j}}) < \sum (\frac{1}{2})^{j+1}$$

Thus we have that: $\prod_{j=1}^{\infty} E_j(\frac{a_j - \hat{a_j}}{z - \hat{a_j}})$ converges uniformly on K. □

Corollary 8.4. For all U open in $\mathbb{C}$, there exists a holomorphic function f whose natural domain of definition is U (it cannot be extended beyond U).

Proof. This is because $\forall U$, there exists sequences $\{a_n\} \in U$ with no accumulation point in U and every point of ∂U is an accumulation point. □

This is a powerful lemma, but for the unit disk think of $a_j = r_j e^{2\pi i j\alpha}$, such that α is irrational and $r_j \to 1$. It forms a spiral, and we use that $\{j\alpha\}$ is dense in [0,1].

If f extends to $\hat{U}$ as $p \in \hat{U}$ accumulation point of zeroes, then $f|_{\hat{U}} \equiv 0 \implies f|_{\hat{U}\cap D(0,1)} \equiv 0 \implies f|_{D(0,1)} \equiv 0$.

What about polar parts (=principle parts)?
Recall that polar/principal parts at α is an expression $s = \sum_{i=-N}^{-1} a_i(z-\alpha)^i = \frac{a_{-1}}{z-\alpha} + \frac{a_{-2}}{(z-\alpha)^2} + \dots$.

Given a sequence of principle points $s_j = \sum_{i=-N}^{-1} a_{ij}(z-\alpha_j)^i$, $\alpha_j \in U \subset \mathbb{C}$ where U is open, no accumulation point α_j in U, is there a function with this on the principle part? Does there exist a function $f : U \to \mathbb{P}^1$ with these principle parts? Yes:

Theorem 8.8. (Miltag-Leffler) There exists a function $f : U \to \mathbb{P}^1$ with principle parts s_j at α_j.

Proof.

1. If there are finitely many s_j's, we set $f = \sum_{j=1}^{k} s_j = \sum_{j=1}^{k}\sum_{i=-N}^{-1} a_{j_i}(z-\alpha_j)^i$.
2. If there are infinitely many s_j's, we again need to flip the function in our earlier proof above.

Lemma 8.4. (Pole Parting) Let $\alpha, \beta \in \mathbb{C}$, $A(z) = \sum_{i=-M}^{-1} a_j(z-\alpha)^j, \quad r < d(\alpha,\beta)$. Then there exists a finite Laurent expansion at β, denoted:

$$B(z) = \sum_{i=-N}^{-1} b_i(z-\beta)^i, \quad k \geq -1$$

such that $|A(z) - B(z)| < \epsilon$ for all $z \in \mathbb{P}^1 \setminus D(\beta, r)$.

Proof. Take the uniformly convergent Laurent expansion on $|z-\beta| \geq R$, truncate it. □

We now continue the proof of our theorem. We can approximate s_j by

$$t_j = \sum_{j=-M_j}^{k} b_{ij}(z-\alpha_j)^j$$

on $|z-\alpha_j| \geq 2d_j$. We then have that $|s_j - t_j| < 2^{-j}$, so on $K \subset \hat{U}$, we have that $\sum_j s_j - t_j$ converges uniformly because $K \cap D(\hat{\alpha_{ji}}, 2d_j) = \emptyset$ for j large. This implues that on K, $|s_j - t_j| < 2^{-j}$. At α_j, the principle part of f is s_j. □

Remark 8.3. This process introduces extra poles, which is why we can't do this on $\mathbb{C}$, only on U.

Remark 8.4. Two normalizations for the Weierstrauss Function:

1. $\zeta(z+1) - \zeta(z) = \eta_1$
2. $\zeta(z+1) - \zeta(z) = 2\hat{\eta}_1$

9 The Prime Number Theorem

9.1 Gamma Function and Riemann Zeta Function

9.1.1 Gamma Function

We will begin with the Gamma Function: For $Re(z) > 0$, at

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

for real valued t.

(1) Does the integral make sense? Near $t = 0$: $t^{z-1} = \exp((z-1)\log(t)) = \exp(Re(z-1) + iIm(z))\log(t) = \exp(Re(z-1)\log(t))\exp(iIm(z)\log(t))$. Note the imaginary part is a phase with norm 1. So we have that this is equal to just $t^{Re(z)-1}$. But we have that:

$$\int_0^t t^\alpha = \frac{t^{\alpha+1}}{\alpha+1}|_0^t$$

for $\alpha > -1$.

Near $t = \infty$: We have that $t^{Re(z)-1}$ given polynomially $\implies t^{Re(z)-1} < e^{t/2}$ for $t >> 0$. This implies that

$$|t^{z-1}e^{-t} \leq e^{t/2}2e^{-t} = e^{-t/2}$$

Proposition 9.1. Let $Re(z) > 0$, then we have that $\Gamma(z+1) = z\Gamma(z)$.

Proof.

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty \underset{f}{t^z} \underset{g'}{e^{-t}} dt \\ &= -t^z e^{-t}|_0^\infty + \int_0^\infty z t^{z-1} e^{-t} dt \\ &= 0 + z\Gamma(z)\end{aligned}$$

□

Corollary 9.1. $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.

Proof. $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$. $\Gamma(2) = 1\Gamma(1)$. $\Gamma(3) = 2\Gamma(2) = 2$. By induction we have the result. □

We will now build up the Gamma Function on $\mathbb{C}$:

1. So we have that $\frac{\Gamma(z+1)}{z} = \Gamma(z)$ is true on $Re(z) > 0$.
2. So $\Gamma(z)$ given $Re(z) > 0$ is defined by $\int_0^\infty \dots$ and $\Gamma(z) = \frac{\Gamma(z+1)}{z}$.
3. But then use $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ to define Γ on $Re(z) \in (-1, 0]$.
4. So it is meromorphic on $Re(z) > -1$ with a simple pole of Residue $\Gamma(1) = 1$.
5. We can repeat this argument: Use $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ to define Γ on $Re(z) > -2$.
6. Then the pole is at $z = -1$ with residue $\Gamma(z) = \frac{1}{z}\Gamma(z+1) = \frac{1}{z}\frac{1}{z+1}\Gamma(z+2)$.
7. We continue to get $\Gamma(z)$ on $\mathbb{C}$ with simple poles at every non-positive integer $(0, -1, -2, \dots)$.
8. Then Residue at $z = -k$ gives:

$$\begin{aligned}\text{Residue} &= \lim_{z\to -k}(z+k)\Gamma(z) \\ &= \lim_{z\to -k}(z+k)\frac{\Gamma(z+1)}{z} \\ &= \lim_{z\to -k}(z+k)(\frac{1}{z}\frac{1}{z+1}\cdots\frac{1}{z+k-1}\frac{1}{z+k})\Gamma(z+k+1) \\ &= \frac{1}{-k}\frac{1}{-k+1}\cdots\frac{1}{-1}\Gamma(1) \\ &= \frac{(-1)^k}{k!}\end{aligned}$$

9.1.2 Riemann Zeta Function

Now we will look at the Riemann Zeta Function:
We are in the situation $Re(z) > 1$. Set $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} e^{-z\log n} = \sum_{n=1}^{\infty} e^{-Re(z)\log(n)} e^{-iIm(z)\log(n)} = n^{-Re(z)}$.

By the comparison test, we know that $\sum_{n=1}^{\infty} n^{-\alpha}$ converges for $\alpha > 1$. This means that $\zeta(z)$ is well-defined.

Idea of the Euler Product: $\frac{1}{1-p^s} = 1 + p^{-s} + p^{-2s} + \ldots$.

Now for all primes and $s < -1$:

$$\frac{1}{1-2^s}\frac{1}{(1-3^s)}\ldots = (1 + 2^s + 2^{2s} + \ldots)(1 + 3^s + 3^{2s} + \ldots)\ldots = 1 + 2^s + 3^s + \ldots$$

We should have that:

$$\frac{1}{\prod(1-p^{-s})} = \zeta(s) \quad \text{for p primes}$$

Consider the product $\prod\limits_{p \text{ prime}} (1-p^z)$ for $Re(z) > 1$. We have that this converges: If $\sum_p |\frac{1}{p^z}| < \infty$ but $\sum_p |\frac{1}{p^z}| \leq \sum_n |\frac{1}{n^z}| < \infty$ for $Re(z) > 1$.

But then we have that: $\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \ldots$.

Now we check: $\frac{1}{2^z}\zeta(z) = \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{6^z} + \ldots$

So we have: $(1 - \frac{1}{2^z})\zeta(z) = \frac{1}{3^z} + \frac{1}{5^z} + \ldots$

Now: $\frac{1}{3^z}(1-\frac{1}{2^z})\zeta(z) = \frac{1}{3^z} + \frac{1}{9^z} + \ldots$

Then: $(1-\frac{1}{3^z})(1-\frac{1}{2^z})\zeta(z) = 1 + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{11^z} + \ldots$

Theorem 9.1. (Sieve of Eratosthenes) $\prod_{p \leq p_N}(1-\frac{1}{p^z})\zeta(z) = 1 + \frac{1}{p_{N+1}z} + \ldots$

$|\prod_{p\leq p_N}(1-\frac{1}{p^z})\zeta(z)| \leq 1 + |\frac{1}{p_{N+1}z}| + |\frac{1}{(p_{N+1})^z}| + \ldots = \sum\limits_{N=P_{N+1}}^{\infty} |\frac{1}{N^z}| \xrightarrow{n\to\infty} 0$ for uniformly, for $z \in K$ compact.

9.2 Prime Number Theorem

Linking $\zeta(z)$ and $\Gamma(z)$.

Proposition 9.2. $\zeta(z) = \frac{1}{\Gamma(z)}\int_0^{\infty} \frac{t^{z-1}e^{-t}}{1-e^{-t}}dt$ for $Re(z) > 1$.

Proof. Integral converges:

(i) Now we look at $t = 0$: $\frac{t^{z-1}e^{-t}}{1-e^{-t}}$= in norm $\frac{t^{Re(z)-1}e^{-t}}{1-e^{-t}} \sim t^{-1+\epsilon} < \infty$.

(ii) At $t = \infty$: t^{z-1} has polynomial growth so $|t^{z-1}| < e^{-t/2}$. So $\frac{e^{-t}}{1-e^{-t}} = \frac{1}{e^t-1} \leq ke^-t$ is also finite.

Have $j \in \mathbb{N}^+$, $j^{-z}(\int_0^{\infty}\tau^{z-1}e^{-\tau}d\tau) = \int_0^{\infty}\frac{1}{j}(\frac{\tau}{j})^{z-1}e^{-\tau}d\tau$

We can set $jt = \tau$, so $jdt = d\tau$, and $\frac{\tau}{j} = t$. So we get:

$$\int_0^{\infty} t^{z-1}e^{-jt}dt$$

So we have that: $j^{-z} = \frac{1}{\Gamma(z)}\int_0^{\infty} t^{z-1}e^{-jt}dt$ So

$$\sum j^{-z} = \frac{1}{\Gamma(z)}\int_0^{\infty}\frac{t^{z-1}}{1-e^{-t}}dt$$

as desired. □

Theorem 9.2. $\zeta(z) - \frac{1}{z-1}$ extends to $Re(z) > 0$.

Proof. For $Re(z) > 1$, we have that $\zeta(z) - \frac{1}{z-1} = \sum_{n=1}^{\infty} \frac{1}{n^z} - \int_0^{\infty} \frac{1}{x^z} dx$. We then have that:

$$\sum_{n=1}^{\infty} (\int_n^{n+1} (\frac{1}{n^z} - \frac{1}{x^z}) dx)$$

Then we can write this as:

$$\sum_{n=1}^{\infty} (\int_n^{n+1} dx (\int_n^x \frac{z du}{u^{z+1}}))$$

This is bounded by its max. So we have that in norm:

$$\leq \sum_{n=1}^{\infty} \frac{|z|}{n^{Re(z)+1}}$$

This +1 lets us work with $Re(z) > 0$ now. □

Remark 9.1. This formula extends ζ to $Re(z) > 0$ with one simple pole at $z = 1$ with residue 1.

Remark 9.2. You can extend it to the whole plane by the $\zeta(1-z) = 2\zeta(z)\Gamma(z)\cos(\frac{\pi}{2}z)2\pi^{-z}$

9.2.1 Things to Keep in Mind for the Following Construction

1. $\zeta(1-z) = 2\zeta(z)\Gamma(z)\cos(\frac{\pi}{2}z)2\pi^{-z}$
2. $Re(z) > 1$: Euler product implies that $\zeta \neq 0$
3. $Re(z) < 0$: Zeroes at $z = -2n$ for $n \in \mathbb{N}$
4. $z = 1$: pole and residue -1
5. $z = 0$: no pole
6. $Re(z) \in [0, 1]$: Need $\zeta \neq 0$ on $Re(z) = 1$

9.2.2 Two Counting Functions

1. $\pi(x)$ is the number of primes $p \leq x$
2. $\Theta(x) = \sum_{p \in [2,x]} \log(p)$

Theorem 9.3. Asymptotically, $\pi(x) = \frac{x}{\log(x)}$. This means $\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log(x)}} = 0$.

Proposition 9.3. $\Theta(x) \sim x \iff \pi(x) \sim \frac{x}{\log(x)}$.

Proof. $\Theta(x) = \sum_{p \leq x} \log(p) \leq \sum_{p \leq x} \log(x) = \pi(x)\log(x)$. Then we have:

$$\frac{\Theta(x)}{x} < \pi(x)\frac{\log(x)}{x}$$

Fix $\epsilon > 0$, then

$$\Theta(x) \geq \sum_{p \in [x^{1-\epsilon}, x]} \log(p) \geq \sum_{p \in [x^{1-\epsilon}, x]} (1-\epsilon)\log(x)$$
$$\geq (1-\epsilon)\log(x)(\pi(x) - \pi(x^{1-\epsilon}))$$

Now for x large, we have:

$$\frac{1}{x}\pi(x)\log(x) \geq (1-\epsilon)\frac{\log(x)\pi(x)}{x} - (1-\epsilon)\underbrace{\pi(x^{1-\epsilon})\frac{\log(x)}{x}}_{\leq x^{1-\epsilon}\frac{\log(x)}{x}\to 0}$$

So $\forall \epsilon > 0$, we have that $\frac{\Theta(x)}{x} \geq (1-\epsilon)\pi(x)\frac{\log(x)}{x}$.
Thus we have our desired result. □

9.2.3 Smoothed out Version

Proposition 9.4. $\Theta(x) \sim x$ if $\lim_{x\to\infty} \int_1^x \frac{\Theta(t)-t}{t^2}dt$ exists.

Proof. Suppose $\Theta(x)$ is too big: $\exists (x_i)$ such that $x_i \to \infty$ with $\Theta(x_i) > \lambda x_i$ such that $\lambda > 1$. Then

$$\int_{x_i}^{\lambda x_i} \frac{\Theta(t)-t}{t^2} \overset{\Theta\text{increasing}}{\geq} \int_{x_i}^{\lambda x_i} \frac{\Theta(x_i)-t}{t^2} \geq \int_{x_i}^{\lambda x_i} \frac{\lambda x_i - t}{t^2}$$

Now by a change of variables $\hat{t} = \frac{t}{x_i}$, we have that:

$$\int_1^{\lambda} \frac{\lambda - t}{t^2} > 0 \quad \text{independent of } x_i$$

This implies that $\int_{x_i}^{\lambda x_i} > C \quad \forall x_i$, so it never converges.

Now suppose that $\Theta(x)$ is too small: $\exists (x_i)$ such that $x_i \to \Theta(x_i) < \lambda x_i, \quad \lambda < 1$. Then we look at:

$$\int_{\lambda x_i}^{x_i} \frac{\Theta(t)-t}{t^2}$$

which is negative, so it also doesn't converge. □

Now we set $x = e^t$:

$$\int_0^{\infty} \frac{\Theta(x)-x}{x^2}dx = \int_0^{\infty} \frac{\Theta(e^t)-e^t}{e^{2t}} = \int_0^{\infty} (\Theta(e^t)e^{-t} - 1)dt$$

9.2.4 Key Facts for Two Counting Functions

1. $\pi(x)$ is the number of primes $p \leq x$
2. $\Theta(x) = \sum_{p\in[2,x]} \log(p)$
3. $\Theta(x) \sim x \iff \pi(x) \sim \frac{x}{\log(x)}$
4. $\Theta(x) \sim x$ if $\lim_{x\to\infty} \int_1^x \frac{\Theta(t)-t}{t^2}dt$ exists $\iff \int_0^{\infty}(\Theta(e^t)e^{-t}-1)dt < \infty$

Now we put in a parameter:

Definition 9.1. $\Phi(z) = \sum_{p \text{ prime}} \log(p)p^{-z}$ for $Re(z) > 1$.

Lemma 9.1. Φ converges absolutely and uniformly on $K \subset \{Re(z) > 1\}$ for K compact.

Proof. For $\epsilon > 0$,

$$|\log(p)p^{-z}| \leq |p^{-z+\epsilon}|$$

and

$$\sum_p |p|^{-s} \leq \sum |n|^{-s} < \infty \quad Re(s) > 1$$

□

9.2.5 Link to $\zeta(z)$

$$[\log(\zeta)]' = \frac{\zeta'(z)}{\zeta(z)} = -\sum_p [\log(1-p^{-z})]'$$
$$= \sum_p \frac{(1-p^{-z})'}{(1-p^{-z})}$$
$$= -\sum \frac{\log(p)}{p^z - 1}$$

Then we have that:

$$-\frac{\zeta'}{\zeta} = \sum_p \frac{\log(p)}{p^z(p^z-1)} + \frac{\log(p)}{p^z} = \sum_p \frac{\log(p)}{p^z(p^z-1)} + \Phi(z)$$

Proposition 9.5. $\sum_p \frac{\log(p)}{p^z(p^z-1)}$ is holomorphic on $Re(z) > \frac{1}{2}$.

Proof.

$$\sum_{p>p_0} |\frac{\log(p)}{p^z(p^z-1)}| \le \sum_{p>p_0} \frac{p^\epsilon}{p^{2z}} \le \sum n^{-2Re(z)+\epsilon} < \infty \quad \text{for } Re(z) > \frac{1}{2}$$

□

This implies that for z in a neighborhood of $Re(z) \ge 1$, we have that:

$$\frac{\zeta'}{\zeta} = \Phi(z) + \sum_p \frac{\log(p)}{p^z(p^z-1)}$$

So $\Phi(z)$ has a pole only at $z = 1$ (with Residue =1), and possibly at any zeroes of $\zeta(z)$ along $Re(z) = 1$.

Theorem 9.4. $\zeta(z) \neq 0$ for $z = 1 + i\alpha$, $\alpha \in \mathbb{R}$.

Proof. Set $\mu(\alpha)$ to be the order of zero at $1+i\alpha$. So $\mu(\alpha) = 0 \implies \zeta(1+i\alpha) \neq 0$. And $\nu(\alpha)$ is the order of zero at $1+2i\alpha$. $\mu(\alpha) = \mu(-\alpha)$ because $\overline{\zeta(\overline{z})} = \zeta(z)$ since its real. This implies that $\frac{\zeta'}{\zeta}$ has a simple pole, residue μ at $1+i\alpha$ and a residue ν at $1+2i\alpha$. Which means that the residue of Φ will be $-\mu$ and $-\nu$. So we get:

$$\lim_{\epsilon\to 0} \Phi(1+\epsilon) = 1$$
$$\lim_{\epsilon\to 0} \Phi(1+\epsilon \pm ix) = -\mu$$
$$\lim_{\epsilon\to 0} \Phi(1+\epsilon \pm 2ix) = -\nu$$

Now we consider the sum:

$$\sum_p \frac{\log(p)}{p(1+\epsilon)}(p^{\frac{i\alpha}{2}} + p^{\frac{-i\alpha}{2}})^4 > 0$$

We have that $p^{\frac{i\alpha}{2}} = e^{i\alpha/2\log(p)}$ is a phase, similarily for other part. Both sum to a real positive value.

$$\sum_p \frac{\log(p)}{p(1+\epsilon)}(p^{\frac{i\alpha}{2}} + p^{\frac{-i\alpha}{2}})^4 = \sum_p \frac{\log(p)}{p^{1+\epsilon}}\left(p^{2i\alpha} + 4p^{i\alpha} + 6 + 4p^{-i\alpha} + 2p^{-2i\alpha}\right)$$
$$= \Phi(1+\epsilon-2i\alpha) + 4\Phi(1+\epsilon-i\alpha) + 6\Phi(1+\epsilon) + 4\Phi(1+\epsilon+i\alpha) + 2\Phi(1+\epsilon+2i\alpha)$$
$$= -\nu - 4\mu + 6 - 4\mu - \nu \quad \text{taking } \epsilon, x \text{ limits}$$
$$= 6 - 8\mu - 2\nu$$
$$> 0$$

Thus Φ has no poles at $1+i\alpha$, $\alpha \neq 0$. □

Lemma 9.2. Φ is almost a Laplace Transform of Θ. That is, $\Phi(z) = z\int_0^\infty e^{-zt}\Theta(e^t)dt$.

Proof. Set $x = e^t$. We have

$$\begin{aligned} z\int_1^\infty \Theta(x)x^{-z}\frac{dx}{x} &= z\int_1^\infty (\sum_{p\leq x}\log(p)x^{-z}\frac{dx}{x}) \\ &= z\sum_p \int_p^\infty \frac{\log(p)}{x^{z+1}}dx \\ &= z\sum_p \frac{\log(p)}{p^{z+1}} \\ &= \Phi(z) \end{aligned}$$

□

Look now at $\int_0^\infty(\Theta(t)e^{-t}-1)e^{-zt}dt$. This becomes:

$$\int_0^\infty \Theta(t)e^{-(z+1)t} - e^{-zt}dt = \frac{1}{z+1}\Phi(z+1) - \frac{1}{z}$$

$\Phi(z+1)$ has a pole at $z=0$, and $\frac{1}{z}$ has a pole at $z=0$, so they cancel out. We just extended, $\Phi(z)$ is ok, holomorphic on a neighborhood of $|Re(z)| \geq 1$, so this implies that: $\int_0^\infty(\Theta(t)e^{-t}-1)e^{-zt}dt$ is holomorphic on a neighborhood of $Re(z) \geq 0$.

But $\frac{1}{z+1}\Phi(z+1) - \frac{1}{z}$ is ok on a neighborhood of $Re(z) \geq 1$.

So if limit $z \to 0$ of $\int_0^\infty(\Theta(t)e^{-t}-1)e^{-zt}dt$ is $\int_0^\infty(\Theta(t)e^{-t}-1)dt$ then we are done.

Theorem 9.5. If $\mathcal{L}(f(t))$ is ok on $Re(z) > 0$ and extends to $U \supset Re(z) \geq 0$ then $\lim_{t\to 0}\mathcal{L}(f(t))$ is the extension.

10 Conclusion

The Steps We Took This Course:

(1) We began with defining F a holomorphic function
   - (i) $[DF, I] = 0$
   - (ii) Cauchy Riemann Equations
   - (iii) $\frac{d}{dz}f = 0$

(2) We then looked at Green's
   - (i) $\int df = \int_{\partial U} f$
   - (ii) Holomorphic: Green works for $f(z) = \frac{1}{z}$

(3) We looked at $-\int_{z_0}^{z} f(\zeta)d\zeta$ well-defined, fundamental theorem of $\mathbb{C}$-calculus

(4) We look at Cauchy Theorem (disk): $f(z) = \frac{1}{2\pi i}\oint_{\partial U}\frac{f(\zeta)}{\zeta - z}d\zeta$

(5) Morera's Theorem (inverse version of Cauchy's Theorem)

(6) Analyticity (expand $\frac{1}{\zeta - z}$)

(7) Cauchy for derivatives: $f^{(n)}(z) = \frac{n!}{2\pi i} \int \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$

(8) Cauchy Estimates: $|f^{(n)}(z)| \leq \frac{cn! \max_{|\zeta - z| = R} |f(\zeta)|}{R^{N+1}}$

(9) Liouville's Theorem and Liouville's for polynomials

(10) Fundamental Theorem of Algebra

(11) Open-Mapping Theorem, Maximum Principle

(12) Normal Form $f'(z) = z^n$

(13) Removeable Singularities

(14) Classification of Singularities

   (i) Removeable singularities
   (ii) Poles $|f(z)| \to \infty$
   (iii) Essential singularities (Thm: A non-constant entire function admits at most two values. The exponential is the example.)

(15) Laurent Series (punctured disk or annulus)

(16) Residues and the Calculus of Residues

(17) Argument Principle $\oint \frac{f'}{f}$ and Rouche's Theorem

(18) Principle Parts

(19) Riemann Surfaces:

   (i) $\mathbb{P}^1 \to \mathbb{C} \cup \{\infty\}$
   (ii) $\mathbb{P}^1 \to \mathbb{P}(\mathbb{C}^2)$
   (iii) Steoreographic Projection $S^3 \to \mathbb{R}^3$ and $S^2 \setminus N \to \mathbb{C}$ for $N \subset S^2$.
   (iv) Elliptic curves and the torus $\mathbb{C}/\tau_1\mathbb{Z} + \tau_2\mathbb{Z} \simeq \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$

(20) Maps:

   (a) $\mathbb{P}^1 \to \mathbb{P}^1$: holomorphic rational maps $\frac{p(z)}{q(z)}$, p, q polynomials, more generally $f : U \to \mathbb{C}$ meromorphic is equivalent to $f : U \to \mathbb{P}^1$ holomorphic.
   (b) Bijective holomorphic maps: $\mathbb{P}^1 \to \mathbb{P}^1$ given by $\frac{az+b}{cz+d}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$
   (c) $\mathbb{C} \to \mathbb{C}$ by $z \to az + b$
   (d) $D \to D$ by $z \to e^{i\theta}(\frac{z-a}{1-\bar{a}z})$
   (e) Upper half plane to upper half plane by $z \to \frac{az+b}{cz+d}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})$

(21) Classification of open simply connected sets U in $\mathbb{C}$

   (i) Either U is $\mathbb{C}$
   (ii) $U \neq \mathbb{C} \implies U \simeq D(0, 1)$ Riemann's Theorem

(22) Elliptic Functions

(i) $f : \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \to \mathbb{C}$ is holomorphic implies its constant

(ii) $f : \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \to \mathbb{P}^1$ is such that number of zeroes is equal to number of poles, sum of the residues is zero, and Abel's Theorem (Sum of locations of zeroes - Sum of locations of poles is in the lattice $\mathbb{Z} + \tau\mathbb{Z}$)

(iii) Weierstrauss $\wp$ function, $\wp'$, ζ, σ

(23) Products

(i) Convergence

(ii) $\prod(1 + |a_i|)$ converges $\iff \sum |a_i|$ converges

(iii) $\prod(1 + a_i)$ converges

(24) ζ, Γ

(25) Prime Number Theorem

Final Notes about Riemann Surfaces:(Not examinable)

1. genus: g is the number of holes
2. $g = 0$: is $\mathbb{P}^1$ and is unique
3. $g = 1$: 1 parameter family $\mathbb{C}/\tau_1\mathbb{Z} + \tau_2\mathbb{Z} \to \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$
4. Theorem: different choices of generators means that $\tau' = \frac{a\tau+b}{c\tau+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$
5. $Im(z) > 0/SL(2, \mathbb{Z})$ has some nice properties
6. $g > 0$: $3g - 3$ parameters, we still don't know what this space looks like (it is not compact)
7. $\Sigma^g \to \mathbb{P}^1$ classify by number of poles

   (i) Fix D: formal sum of points

   (ii) we want $f \to$ formal sum of poles of f

   (iii) This is by Abel's Theorem: a g-dimensional space of holomorphic differentials $\{w_1, \ldots, w_g\}$ and there exists a function f with poles at p_i, zeroes at $q_i \iff$ there exists a cycle C on Σ (a cycle is a formal sum of closed loops) with $\sum_i \int_{q^i}^{p_i}(w_1, \ldots, w_g) = \int_C (w_1, \ldots, w_g)$. Elliptic $w = dz$.
8. Abel map $\Sigma^g \to \mathbb{C}^g$ by $z \mapsto \int_{z_0}^{z}(w_1, \ldots, w_g)$ defined mod integrals of cycles: $\mathbb{C}^g/\mathbb{Z}^g + \Lambda\mathbb{Z}^g$.
9. $g = 1/\mathbb{Q}$ by $g^2 = ax^3 + bx + c$ can link this to modular forms.

Remark 10.1. Professor Hurtubise is retiring after this year. He has been a great professor and this was the most fun I have had in a math course at McGill University. I wish him the best in his retirement.