MATH 255 - Honours Analysis II

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Contents

1 Topology of Metric Spaces

Lecture 1 - 01/07/2025

Objective: *Extend the analysis of $\mathbb{R}$ to other spaces*

1 Distance functions and metric spaces

Definition 1.1: Distance functions

Given a set X for which we have $d : X \times X$, we say that a function $d : (x, y) \mapsto d(x, y) \qquad X \times X \to [0, \infty)$ is a distance function on a metric if:

- $\forall (x, y) \in X \times X \qquad d(x,y) \begin{cases} is > 0 & \text{if } x \neq y \\ = 0 & \text{if } x = y \end{cases}$, i.e. $d(x,y) = 0 \iff x = y$

- $\forall (x, y) \in X \times X \qquad d(x, y) = d(y, x)$ symmetry

- $\forall x, y, z \in X \qquad d(x, z) \leq d(x, y) + d(y, z)$ Triangle inequality

Any set X equipped with a metric d is called a **metric space.**

Example 1.1

(i) For every $n \in \mathbb{N}$ and $p \in [1, \infty]$, with say $\mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$, we have

$$d_p : (x, y) \mapsto \begin{cases} \left(\sum_{i=1}^{n} |x_1 - y_i|^p \right)^{1/p} & \text{if } 1 \leq p \leq \infty \\ max\{1 \leq i \leq n\} \ |x_i - y_i| & \text{if } p = \infty \end{cases}$$

We also denote $d_p(x, y) = ||x - y||_p$ or simplify $||x - y||$ if $p = 2$. The distance d_2^p is also called Euclidean distance. d_p is a distance is $\mathbb{R}^n \quad \forall \, p \in [1, \infty]$. The only **non-trivial** point to prove is the triangle inequality; we will prove this later.

Say $p \in (0,1)$, then d_p is **not a distance**. Indeed, $d_p\Big((1,0,\dots,0),(0,1,0,\dots,0)\Big) = (1+1)^{1/p} = 2^{1/p} > 2 = d\big((1,0,\dots,0),(0,1,\dots,0)\big) + d\big((0\dots0),(0,1,0,\dots,0)\big)$

Example 1.2

(ii) $\mathbb{C}\times\mathbb{C} \to [0,\infty) \qquad (z_1,z_2) \mapsto |z_1 - z_2| = \sqrt{(y_1-x_1)^2+(y_2-x_2)^2} = d_2\Big((x_1,y_1),(x_2,y_2)\Big)$ if we identify $\mathbb{C}$ with $\mathbb{R}^2$.

1.1 The discrete metric

(iii) Let X be a set. Define $d : X\times X \to [0,\infty) \qquad d:(x,y)\mapsto \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x\neq y \end{cases}$ d is a distance in X called the discrete metric. Proof:

- $\forall x,y\in X \qquad d(x,y)\in\{0,1\} \Rightarrow d(x,y)\geq 0$
- $\forall x,y\in X \qquad d(x,y)=0 \iff x=y$ by definition
- $\forall x,y\in X \qquad d(x,y)=d(y,x)$
- $\forall x,y\in X \qquad d(x,z) = \begin{cases} 0=0+0 & \text{if } x=y=z \\ 1=0+1 & \text{if } x=y\neq z \\ 0<1+0 & \text{if } x=z\neq y \\ 1=1+0 & \text{if } y=z\neq x \\ 1<1+1 & \text{if } x\neq y, y\neq z, z\neq x \end{cases}$

2 Normal Vector Spaces

Definition 2.1: Norm functions

Given a vector space X over $\mathbb{R}$ or $\mathbb{C}$, we say that a function $||\ .\ || : X \to [0,\infty) \qquad x\mapsto ||x||$, with $||x||$ being a **norm** on X is :

- $\forall x \in X, \quad ||x|| \begin{cases} is \ > 0 & \text{if } x \neq 0 \\ = 0 & \text{if } x = 0 \end{cases}$

- $\forall x \in X \quad \lambda \in \mathbb{R} \vee \mathbb{C}, \qquad ||\lambda x|| = |\lambda| \, ||x||$

- $\forall x, y \in X \qquad ||x + y|| \leq ||x|| + ||y||$ Triangle Inequality

Any vector space X over $\mathbb{R} \vee \mathbb{C}$ equipped with a norm $||\ .\ ||$ is called a normal vector space

Proposition 2.1

If $||\ .\ ||$ is a norm on a vector space X, then $d : X \times X \to [0, \infty) \qquad d : (x, y) \mapsto d(x, y) = ||x - y||$ is a **metric** on X

Proof:

- $d(x, y) = ||x - y|| > 0$ by definition.

- $d(x, y) = 0 \iff ||x - y|| = 0 \iff x - y = 0 \iff x = y \ by \ (i)$

- $d(x, y) = ||x - y|| = ||(-1)(y - x)|| = |-1| \, ||y - x|| = ||y - x|| = d(y, x) by \ (ii)$

- $\forall x, y, z \in X \qquad d(x, z) = ||x - z|| = ||(x - y) + (y - z)|| \leq ||x - y|| + ||y - z|| = d(x, y) + d(y, z)$

$\square$

Example 2.1

- in $\mathbb{R}^n, \quad ||\ .\ ||_p$

- Let $[a, b]$ be a compact interval in $\mathbb{R}$ and $C\big([a, b]\big)$ be the set of all continuous functions on $[a, b]$. $C\big([a, b]\big)$ is a vector space. Define $||\ .\ ||_\infty \qquad C\big([a, b]\big) \to [0, \infty) \qquad f \mapsto ||f||_\infty \quad = max_{x \in [a,b]} \, |f(x)|$

$||\ .\ ||_\infty$ is a norm on $C\big([a, b]\big)$ called the uniform norm.

Proof;

- $||f||_\infty \geq 0$ since $|f(x)| \geq 0 \quad \forall\, x \in [a,b]$

- $||f||_\infty = 0 \Rightarrow max_{x\in[a,b]}\ |f(x)| = 0 \Rightarrow \forall\ x \in [a,b], \quad |f(x)| \leq 0 \Rightarrow \forall\ x \in [a,b], \quad f(x) = 0$, i.e. $f = 0$. Conversely, $f = 0 \Rightarrow ||f||_\infty = ||0||_\infty = max_{x\in[a,b]}\ |0| = 0$.

- $\forall\ f \in C\big([a,b]\big), \lambda \in \mathbb{R} \qquad ||\lambda f||_\infty = max \frac{|\lambda f(x)|}{|\lambda|\,|f(x)|} = |\lambda|\ max|f(x)| = |\lambda|\ ||f||_\infty = |\lambda|\ |f(x)|$

Lecture 2 - 01/09/2025

3 Hölder's Inequality

For all $p \in [1,\infty], \quad x, y \in \mathbb{R}^n$, we have

$$\sum_{i=1}^{n} |x_i y_i| \leq ||x||_p ||y||_{p'}$$

, where $p' \in [1,\infty]$ is such that $\frac{1}{p} + \frac{1}{p'} = 1$, with the convention that $\frac{1}{\infty} = 0$. i.e. $p' = \begin{cases} \frac{p}{p-1} & \text{if } 1 < p < \infty \\ 1 & \text{if } p = \infty \\ \infty & \text{if } p = 1 \end{cases}$

Proof:

Case $p = \infty$

$\forall\ i \in \{1,\ldots,m\}, \quad |x_i y_i| = |x_i|\ |y_i| \leq \big(max_{i\leq j\leq m}\ |x_j|\big)|y - i| = ||x||_\infty$. Hence,

$$\sum_{i=1}^{n} |x_i y_i| \leq \sum_{i=1}^{n} ||x||_\infty\ |y_i| = ||x||_\infty\ ||y||_1$$

Remark 3.1

Pretty much the same for case of $p = 1$, simply reversed.

Case $1 < p < \infty$

First assume that $x \neq 0$ and $y \neq 0$ (otherwise, proof is trivial).
Now let $x' = \frac{x}{||x||_p}$, $\quad y' = \frac{y}{||y||_{p'}}$ and observe that $||x'||_p = ||\frac{x}{||x||_p}|| = \frac{||x||_p}{||x||_p} = 1$ and $||y'||_{p'} = ||\frac{y}{||y||_{p'}}||_{p'} = \frac{||y||_{p'}}{||y||_{p'}} = 1$, so that it follows that

$$\sum_{i=1}^{n} |x_i' y_i'| = \sum_{i=1}^{n} \frac{|x_i y_i|}{||x||_p \; ||y||_{p'}}$$

, so that $\sum |x_i y_i| \leq |||x||_p \; ||y||_{p'}$ iff $\sum |x_i' y_i'| \leq 1$

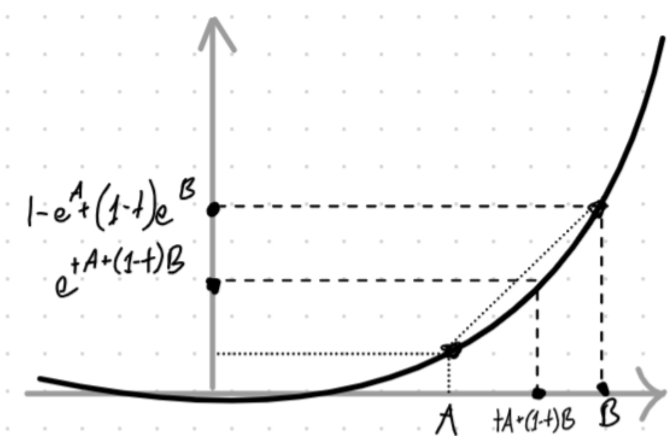
3.1 Young's Inequality

Use Young's Inequality, $\forall\, a, b \geq 0, \quad ab \leq \frac{1}{p} a^p + \frac{1}{p'} b^{\,p'}$, which follows from the convexity of exponential.

Proof of Young's Inequality

$$ab = e^{ln\, a} \cdot e^{ln\, b} = e^{1/p \cdot ln\; a^p} \cdot e^{1/p' \cdot ln\, b^{\,p'}} = e^{1/p \cdot ln\, a^p + 1/p' \cdot ln\, b^{\,p'}} = e^{tA + (1-t)B}$$

,where $t = \frac{1}{p}$, $\quad 1 - t = 1 - \frac{1}{p} = \frac{1}{p'}$, $\quad A = ln\, a^p$, $\quad B = ln\, b^{\,p'}$ With $e^{tA+(1-t)B} \leq te^A + (1-t)e^B$ which



follows from the fact that $(\frac{d}{dx})^2(e^x) = e^x > 0$

Remark 3.2

Let's assume this fact for the moment: we will study the exponential and its properties in the last part of the course.

It follows that $ab \leq te^A + (1-t)e^B = te^{ln\, a^p} + (1-t)e^{ln\, b^{\,p'}} = ta^p + (1-t)b^{\,p'}$ $\square$

Note that when $p = 2$, Young's Inequality follows from: $ab = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{2}(a-b)^2 \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$

Now that $a = |x'_i|$ and $b = |y'_i|$ and apply Young's Inequality:

$$|x'_i y'_i| = |x'_i|\,|y'_i| \leq \frac{1}{p}|x'_i|^p + \frac{1}{p'}|y'_i|^{p'} \Rightarrow \sum_{i=1}^{n} |x'_i y'_i| \leq \frac{1}{p}\sum_{i=1}^{n} |x'_i|^p + \frac{1}{p'}\sum_{i=1}^{n} |y'_i|^{p'}$$

from which we get $||x'||_p^p = 1$ and $||y'||_{p'}^{p'} = 1$, bringing us to $\frac{1}{p} + \frac{1}{p'} = 1$.

And so, it follows that

$$\sum_{i=1}^{n} |x_i y_i| \leq ||x||_p \, ||y||_{p'}$$

$\square$

4 Minkowski's Inequality *(triangle inequality)*

For all $p \in [1, \infty], \quad x, y \in \mathbb{R}^n$, we have $||x + y||_p \leq ||x||_p + ||y||_p$

Proof:

Case $p = \infty$

$\forall i \in \{1, \ldots, m\}, \quad |x_i + y_i| \leq |x_i| + |y_i| \leq \left(max_{1\leq j\leq m}|x_j|\right) + \left(max_{1\leq j\leq m}|y_j|\right)$; hence, $||x+y||_\infty = max_{1\leq j\leq m}|x_i + y_i| \leq ||x||_\infty + ||y||_\infty$

Case $p = 1$

$$|x_i + y_i| \leq |x_i| + |y_i| \Rightarrow \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = ||x||_1 \, ||y||_1$$

Case $1 < p < \infty$

$$||x + y||_p^p = \sum_{i=1}^{n} |x_i + y_i|^p = \sum_{i=1}^{n} |x_i + y_1|^{p-1} |x_i + y_i|$$

Remark 4.1

Since we transformed the expression into the sum of a product with $|x_i + y_i|$, we were able to use the triangle inequality proven above!

$$\sum_{i=1}^{n} |x_i + y_i|^{p-1} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i + y_i|^{p-1} |x_i| + \sum_{i=1}^{n} |x_i + y_i|^{p-1} |y_i|$$

By Hölder's Inequality, it follows that

$$||x+y||_p^p \leq \left(\sum |x_i + y_i|^{(p-1)p'} \right)^{1/p'} \left(\sum |x_i|^p \right)^{1/p} + \left(\sum |x_i + y_i|^{(p-1)p'} \right)^{1/p'} \left(\sum |y_i|^p \right)^{1/p}$$

Consider now that $(p-1)p' = p$, our inequality above now becomes $||x+y||_p^{p/p'} \, ||x||_p + ||x+y||_p^{p/p'} \, ||y||_p = ||x+y||_p^{p/p'}(||x||_p + ||y||_p)$, which gives

$$||x+y||_p^{p-p/p'} \leq ||x||_p + ||y||_p, \quad \text{with } p - (p-1) = 1$$

□

Proposition 4.1

For every $p \in [1, \infty]$, $\;||.||_p$ is a **norm in** $\mathbb{R}^n$

Proof.

- $||x||_p \geq 0$ by definition ✓
- $||x||_p = 0 \iff \forall\, i \in \{1, \dots, m\}, \quad |x_i| = 0$, since $|x_i| \geq 0$, $\iff \forall\, i \in \{1, \dots, m\}, \quad x_i = 0$
- $\forall\, \lambda \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad ||\lambda x||_p = \begin{cases} \left(\sum |\lambda x|^p \right)^{1/p} = |\lambda| \left(\sum |x_i|^p \right)^{1/p} = |\lambda| \, ||x||_p & \text{if } 1 < p < \infty \end{cases}$

□

Lecture 3 - 01/14/2025

5 Inner product spaces

Inner product spaces are an important claim of normed vector spaces.

Definition 5.1: Inner product spaces

Given a vector space X over $\mathbb{R}$, we say that a function $\langle .,. \rangle : X \times X \to \mathbb{R}, \quad (x, y) \mapsto \langle x, y \rangle$, such that it satisfies:

1. $\forall x \in X, \quad \langle x, x \rangle \geq 0$
2. $\forall x \in X, \quad \langle x, x \rangle = 0 \iff x = 0$

3. $\forall x, y \in X \quad \langle x, y\rangle = \langle y, x\rangle$ symmetry

4. $\forall x, y, z \in X$ and for $\lambda, \mu \in \mathbb{R}$, $\langle x, \lambda y + \mu z\rangle = \lambda\langle x, y\rangle + \mu\langle x, z\rangle$ Linearity with respect to y

, is an inner product in X.

Any vector space X over $\mathbb{R}$ equipped with an inner product $\langle ., .\rangle$ is called an inner product space

Remark 5.1

1. From the items 3 and 4 above, it follows that $\langle \lambda x + \mu z,\ y\rangle = \lambda\langle x, y\rangle + \mu\langle z, y\rangle$ represents linearity with respect to x, but because of symmetry, it is enough to prove linearity with respect to only one variable.

2. The definition can be extended to the case of vector spaces over $\mathbb{C}$ by replacing item 3 (symmetry) by $3'$: $\forall\ x, y \in X,\ \ \langle y, x\rangle = \overline{\langle x, y\rangle}$. In this case, $3'$ and 4 give $\langle \lambda x + \mu z, y\rangle = \overline{\lambda}\langle x, y\rangle + \overline{\mu}\langle z, y\rangle$ (**anti-linearity** with respect to x), and such $\forall x, y, z, \lambda, \mu \in \mathbb{C}$.

Example 5.1

Let $A = (a_{ij})_{1\leq i,j\leq n}$ be a $n \times n$ matrix with real coefficients. Define $\langle ., .\rangle_A : \ \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (x, y) \mapsto \langle x, y\rangle_A = \sum_{i,j=1}^{n} a_{ij}x_iy_j$.

Assume that A is symmetric ($a_{ij} = a_{ji}\ \ \forall i, j \in \{1, \ldots, n\}$) and A is definite positive, i.e. $\forall x \in \mathbb{R}^n, \ \ \langle x, x\rangle_A \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$.

Then $\langle ., .\rangle_A$ is an inner product on $\mathbb{R}^n$, since 1,2,3 follow from the assumptions on A and linearity is easy:
$\forall x, y, z \in \mathbb{R}^n$, $\forall \lambda, \mu \in \mathbb{R}$, $\langle x,\ \lambda y + \mu z\rangle_A = \sum_{i,j=1}^{n} x_i(\lambda y_j + \mu z_j) = \lambda\sum_{i,j=1}^{n} x_iy_i + \mu\sum_{i,j=1}^{n} x_iz_j = \lambda\langle x, y\rangle_A + \mu\langle x, z\rangle_A$ and so $\langle x, y\rangle_A = \sum_{i=1}^{n}\lambda_ix_iy_i, \quad \langle x, x\rangle_A = \sum_{i=1}^{n}\lambda_ix_i^2 \geq 0$, or $\langle x, x\rangle_A = 0 \iff x_i = 0 \ \ \forall i$

Now let us be in a 2-dimensional space, such that $n = 2$ and $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ such that:

$$\langle x, y\rangle_A = (x_1, x_2)(y_1, y_2) = ax_1y_1 + bx_2y_2 + cx_1y_2 + cx_2y_1$$
$$\langle x, x\rangle_A = ax_1^2 + bx_2^2 + 2cx_1x_2, \quad \text{with } \langle x, x\rangle_A, \geq 0, \quad = 0 \iff (x_1, x_2) = (0, 0)$$

, and so we have $a > 0$, $\quad ab - c^2 > 0$, with the discriminant$= (cx_2)^2 - a(bx_2^2) = x_2^2(c^2 - ab)$, with discriminant$\leq 0$.

Remark 5.2

Results from Algebra: Symmetric matrices have real Eigen values. A symmetric matrix is definite positive if and only if its Eigen values are strictly positive (> 0).

6 Cauchy-Schwartz Inequality

Let $(X, \langle ., .\rangle)$ be an inner product space. Then,

$$\forall x, y \in X, \quad |\langle x, y\rangle| \leq \sqrt{\langle x, y\rangle\langle y, y\rangle}.$$

Proof:

Define $f(t) = \langle x + yt,\ x + yt\rangle = \langle x, x\rangle + 2\langle x, y\rangle t + \langle y, y\rangle t^2$ by linearity and symmetry.

Observe that $f(t) \geq 0 \;\; \forall t \in \mathbb{R}$, hence the discriminant must be negative, such that discriminant$=$ $\langle x, y\rangle^2 - \langle x, x\rangle\langle y, y\rangle \leq 0$; i.e. $|\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle\langle y, y\rangle}$, which is the Cauchy-Schwartz inequality. $\square$

Proposition 6.1

Let $(X, \langle ., .\rangle)$ be an inner product space and $||.|| : \quad X \to [0, \infty), \quad x \mapsto \sqrt{\langle x, x\rangle}$. Then, $||.||$ is a norm on X.

Proof:

1. $||x|| - \sqrt{\langle x, x\rangle} \geq 0$, with $||x|| = 0 \iff x = 0$, ✓

2. $\forall x \in X, \quad \forall \lambda \in \mathbb{R}, \quad ||\lambda x|| = \sqrt{\langle \lambda x, \lambda x \rangle} = \sqrt{\lambda \langle \lambda x, x \rangle} = \sqrt{\lambda^2 \langle x, x \rangle} = |\lambda| \sqrt{\langle x, x \rangle} = |\lambda| ||x||$ ✓
(we have linearity with respect to both x's, meaning we take out a λ for each of them, thus we get λ^2 in the square root)

3. $\forall x, y \in X, \quad ||x+y|| = \sqrt{\langle x+y, x+y \rangle} = \sqrt{\langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle} \leq \sqrt{(||x|| + ||y||)^2} \Rightarrow$ $||x||^2 \leq ||x|| \, ||y|| = ||x|| + ||y||$, by Cauchy-Schwartz ✓

□

6.1 Order of spaces

Metric spaces $\supset$ Normed Vector spaces $\supset$ Linear Product spaces

6.2 Open balls

Definition 6.1: Open balls

Let (X, d) be a metric space, $x \in X$ and $r > 0$. We call open ball of centre x and radius r the set

$$B(x, r) = \{y \in X : \quad d(x, y) < r\}, \qquad \text{also denoted } B_d(x, r) \text{ or } B_{(X,d)}(x, r)$$

Example 6.1

$X = \mathbb{R}^n, \quad d(x, y) = ||x - y||_p$, we have

$$B_{d_p}(x, r) = \{y \in \mathbb{R}^n : \begin{cases} \left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p} < r & \text{if } p < \infty \\ max_{i \leq i \leq n} |x_i - y_i| < r & \text{if } p = \infty \end{cases} \}$$

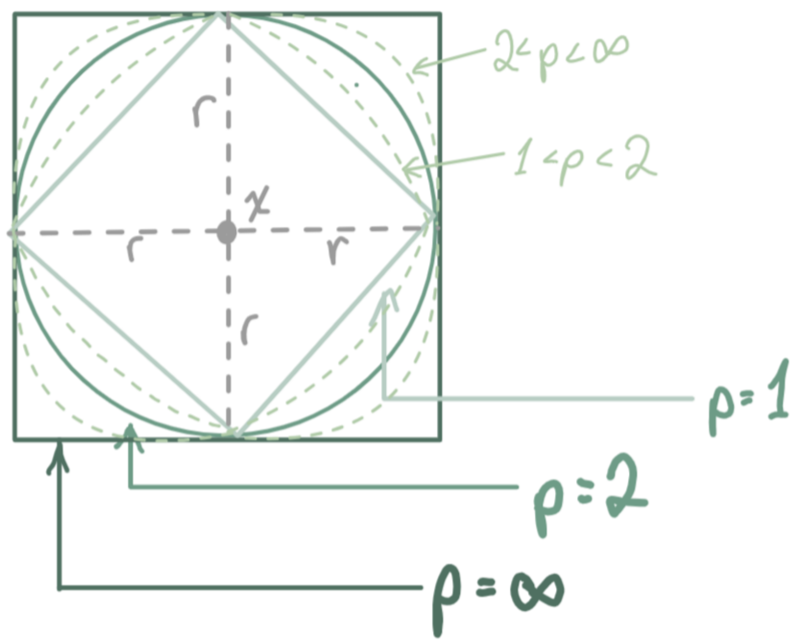
In the case $n = 1$, $\forall p, \quad B_{d_p}(x, r) = (x - r, x + r)$. In this case, every interval (a, b), with $-\infty < a < b < \infty$ is an open ball of centre $\frac{a+b}{2}$ and radius $\frac{b-a}{2}$.

Lecture 4 - 01/16/2025

Definition 6.2: Openness and Closedness

Let (X, d) be a metric space and $A \subseteq X$.

1. We say that A is **open** in (X, d) if $\forall x \in A, \quad \exists r > 0$ such that $B(x, r) \subseteq A$



2. We say that A is **closed** in (X,d) if A's **complement**, $X \backslash A$, is **open** in (X,d)

Proposition 6.2

Let (X,d) be a metric space, $x_0 \in X$, $\quad r_0 > 0$:

1. $B(x_0, r_0)$ is open in (X,d)

2. $\{y \in X : \ d(x_0, y) \leq r_0\}$ is textbfclosed in (X,d)

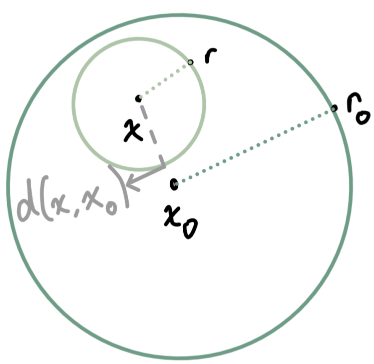


Figure 1: Representation for Proposition 4.2

Proof:

(1) Let $x \in B(x_0, r_0)$ and $r = r_0 - d(x, x_0)$. $\forall y \in B$, $\quad d(x_0, y) \leq d(x_0, x) + d(x, y) \leq r_0$, i.e.

$y \in B(x_0, r_0)$, which proves that $B(x,r) \subseteq B(x_0, r_0)$

(2) Define $A = \{y \in X : \ d(x_0, y) \leq r_0\}$. Let $x \in X \backslash A$ and $r = d(x, x_0) - r_0 \ \ \forall y \in B(x,r)$, then we have:

$$d(x_0, x) \leq d(x_0, y) + d(y, x), \text{ and } d(y,x) < r = d(x, x_0) - r_0 \Rightarrow r_0 < d(x_0, y)$$

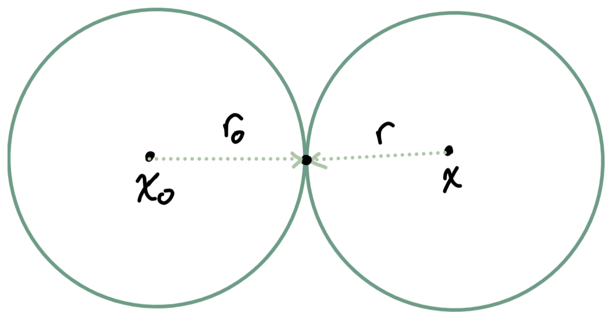


Figure 2: Representation for (2)

This shows that $B(x,r) \subseteq X \backslash A$. Since this is true $\forall x \in X \backslash A$, we obtain that $X \backslash A$ is open in (X,d), i.e. A is closed in (X,d). □

6.3 Properties of sets

Let (X,d) be a metric space:

1. X and $\emptyset$ are **both** open and closed

2. If $(A_i)_{i \in I}$ is a collection of **open** sets in (X,d), then $\bigcup_{i \in I}, A_i$ is an **open** set in (X,d).

3. If $(A_i)_{1 \leq i \leq n}$ is a **finite collection** of **open** sets in (X,d), then $\bigcup_{i \in I}^n A_i$ is **open** in (X,d).

4. If $(A_i)_{i \in I}$ is a collection of **closed** sets in (X,d), then $\bigcap_{i=1}$ is **closed** in (X,d).

5. If $(A_i)_{1 \leq i \leq n}$ is a **finite collection** of **closed** sets in (X,d), then $\bigcup_{i=1}^n A_i$ is **closed** in (X,d)

6. If A_1 is an **open** set in (X, d) and A_2 is a **closed** set in (X, d), then $A_1 \backslash A_2$ is **open** in (X, d)

7. If A_1 is a **closed** set in (X, d) and A_2 is an **open** set in (X, d), then $A_1 \backslash A_2$ is **closed** in (X, d)

Proof. 1. • $\forall x \in X, \quad B(x, 1) \subseteq X$, which is true for every radius; hence, X is open.

• $\emptyset$ is open: trivial since $\nexists\, x \in \emptyset$

• X is closed since $X \backslash X = \emptyset$ is open.

• $\emptyset$ is closed since $X/\emptyset = X$ is open. ✓

2. Let $(A_i)_{i \in I}$ be a collection of open sets in (X, d). Let $x \in \cup_{i \in I} A_i$, i.e. $\exists\, i_x \in I$ such that $x \in A_{i_x}$. Since A_{i_x} is open, $\exists\, r_x > 0$ such that $B(x, r_x) \subseteq A_{i_x}$. Since $A_{i_x} \subseteq \cup_{i \in I} A_i$, it follows that $B(x, r_x) \subseteq \cup_{i \in I} A_i$. ✓

3. Let $(A_i)_{1 \leq i \leq n}$ be a finite collection of open sets in (X, d). Let $x \in \cap_{i=1}^{n} A_i$, i.e. $\forall i \in I, \quad x \in A_i$. Since A_i is open, $\exists\, r_i > 0$ such that $B(x, r_i) \subseteq A_i$. Let $r = min(r_1, \ldots, r_n)$. Then $B(x, r) \subseteq \cap_{i=1}^{n} B(x, r_i) \subseteq \cap_{i=1}^{n} A_i$. ✓

4. Let $(A_i)_{i \in I}$ be a collection of closed sets in (X, d). Write $\cap_{i \in I} A_i = X \backslash \cup_{i \in I} (X \backslash A_i)$ *(by De-Morgan's Law!)*. The sets $X \backslash A_i$ are open since A_i are closed. Using property (2) from above, we get that $\cup(X \backslash A_i)$ is open; hence, $\cap A_i$ is closed. ✓

5. Let $(A_i)_{1 \leq i \leq n}$ be a finite collection of closed sets in (X, d) and $\cup_{i=1}^{n} A_i - X \backslash \cap_{i=1}^{n} (X \backslash A_i)$. The sets $X \backslash A_i$ are open, since A_i are closed. Then, by property (3), given that $\cap_{i=1}^{n} (X \backslash A_i)$ is open, we know that $\cup_{i=1}^{n} A_i$ is closed. ✓

6. Let A_1 be an open set in (X, d), and A_2, a closed set. Then, $A_1 \backslash A_2 = A_1 \cap (X \backslash A_2)$ is open by property (3), since A_1 and $X \backslash A_2$ are both open. ✓

7. Similarly to above. Write again $A_1 \backslash A_2 = A_1 \cap (X \backslash A_2)$ and observe that A_1 and $X \backslash A_2$ are, in this case, both closed. ✓

□

Counter-example 1:

$$\text{In } \mathbb{R}^n, \quad \bigcap_{k=1}^{\infty} B_{d_p}\left(0, \frac{1}{k}\right) = \left\{ x \in \mathbb{R}^n : \ d_p(x,0) < \frac{1}{k}, \quad \forall\, k \right\} = \{0\}$$

Indeed, $\forall\, x \in \mathbb{R}^n$, $[\forall\, k \in \mathbb{N}, \quad ||x||_p < \frac{1}{k}] \Rightarrow ||x||_p \leq \lim \frac{1}{k}$.
Conversely, $x = 0 \Rightarrow \forall\, k \in \mathbb{N}, \quad ||x||_p < \frac{1}{k}$. Moreover, $\{0\}$ is not open since $\forall\, r > 0, \quad B_{d_p}(0, r) \not\subseteq \{0\}$.
Indeed, $\left(\frac{r}{2}, 0 \ldots, 0\right) \in B(0, r) \backslash \{0\}$, since $0 < ||(\frac{r}{2}, 0, \ldots, 0)||_p = \frac{r}{2} < r$

Counter-example 2:

$$\text{In } \mathbb{R}^n, \quad \bigcup_{k=1}^{\infty} (\mathbb{R}^n \backslash B_{d_p}(0, \frac{1}{k})) = \mathbb{R}^n \backslash \bigcap_{k=1}^{\infty} B_{d_p}(0, \frac{1}{k}) = R^n \backslash \{0\}$$

$\mathbb{R}^n \backslash \{0\}$ is not closed, since $\{0\}$ is not open, but the sets $\mathbb{R}^n \backslash B_{d_p}(0 \frac{1}{k})$ are all closed, since $B_{d_p}(0 \frac{1}{k})$ are open.

Lecture 5 - 01/21/2025

Case of the discrete metric $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

In this case, all sets $A \subseteq X$ are open, since $\forall\, x \in A, \quad B(x, 1) = \{x\} \subseteq A$. It follows that all sets are also closed in this case.

Remark 6.1

When several metrics with different open sets are involved, it is important to specify the metric. For example, $\{0\}$ is **not** an open set in $(\mathbb{R}^n, ||.||_p)$, but it **is** an open set in $(\mathbb{R}^n, d)$, where d is the discrete metric.

s

7 Closure, interior, and boundaries of metrics

Definition 7.1: Closure, Interior and Boudaries

Let (X, d) be a metric space and $A \subseteq X$.

1. We call **closure** of A in (X, d) the set $\overline{A}$ of all $x \in X$ such that $\forall\, r > 0, \quad B(x, r) \cap A \neq \emptyset$.

2. We call **interior** of A in (X, d) the set $\mathring{A}$ of all $x \in X$ such that $\exists\, r > 0$ such that $B(x, r) \subseteq A$

3. We call **boundary** of A in (X, d) the set $\partial A = \overline{A} \setminus \mathring{A}$

Example 7.1

1. Case of a normed vector space $(X, ||.||)$

In this case, $\overline{B(x_0, r_0)} = \{x \in X : ||x - x_0|| \leq r_0\}$. Indeed, $\forall x \in X$:

1. if $||x - x_0|| > r_0$, then $B(x,\, d(x, x_0) - r_0) \cap B(x_0, r_0) = \emptyset$ (we proved this earlier). Hence, $x \notin \overline{A}$.

2. if $||x - x_0|| = r_0$, then $\forall \ r > 0$, $B(x, r) \cap B(x_0, r_0) \neq \emptyset$.Indeed, let $x_\varepsilon = (1 - \varepsilon)(x - x_0) + x_0 \in B(x_0, r_0) \cap B(x, r)$, this is true because $||x_\varepsilon - x_0|| = \Big|\Big| \big[(1 - \varepsilon)(x - x_0) + x_0\big] - x_0\Big|\Big| = \Big|\Big|(1 - \varepsilon)(x - x_0)\Big|\Big|$, and considering $\varepsilon > 0 \Rightarrow (1 - \varepsilon) \in (0, 1)$, we know $(1 - \varepsilon)||x - x_0|| < ||x - x_0||$, with $||x - x_0|| = r_0$, such that we get $||x_\varepsilon - x_0|| < r_0$. Also, we have $||x_\varepsilon - x|| = \Big|\Big| \big[(1 - \varepsilon)(x\_x_0) + x_0\big] - x|| = \Big|\Big|(1 - \varepsilon - 1)(x - x_0)\Big|\Big| = \varepsilon||x - x_0|| = \varepsilon r_0 < r$. Since we now have $||x_\varepsilon - x_0| < r_0$ and $||x_\varepsilon - x|| < r$, we have the implication that $x \in \overline{A}$.

3. if $||x - x_0|| < r_0$, then $\forall\, r > 0, \quad x \in B(x, r) \cap B(x_0, r_0)$. Hence, $B(x, r) \cap B(x_0, r_0) \neq \emptyset$, thus implying that $x \in B(x_0, r_0)$

It follows that $B(x_0, r_0) = \{x \in X : ||x - x_0|| \leq r_0\}$. It is not true in general that $\overline{B(x_0, r_0)} = \{x \in X : d(x, x_0) \leq r_0\}$. Consider the counter-example of the discrete metric and $r_0 = 1$.

Then, $\forall x \in X$ and $\forall r \in (0,1)$, $B(x_0, r_0) \cap B(x,r) = \{x_0\} \cap \{x\} \neq \emptyset$ iff $x = x_0$. Hence, $\overline{B(x_0, r_0)} = \{x_0\} \neq \{x \in X : d(x, x_0) \leq 1\} = x$

Example 7.2

2. Case of the normed vector space $(X, ||.||)$.

In this case, $interior\overbrace{\{x \in X : ||x - x_0|| \leq r_0\}}^{\circ} = B(x_0, r_0)$ and let $A = \{x \in X : ||x - x_0|| \leq r_0\} = \overline{B(x_0, r_0)}$.

Indeed, $\forall x \in X$:

1. *(Let $A = \overline{B(x_0, r_0)}$)* if $||x - x_0|| > r_0$, then $\forall r > 0, \quad B(x,r) \not\subseteq A$, since $x \notin A$; hence, $x \notin \mathring{A}$.

2. if $||x - x_0|| = r_0$, then $\forall r > 0, \quad B(x,r) \not\subseteq A$. Consider $x_\varepsilon = (1+\varepsilon)(x - x_0) + x_0$ and proceed similarly to what we did in **Case 1**:

$$||x_\varepsilon - x_0|| = \Big|\Big|\big[(1+\varepsilon)(x - x_0) + x_0\big] - x_0\Big|\Big| = (1+\varepsilon) \cdot r_0 \Rightarrow ||x_\varepsilon - x_0|| > r_o \quad (\text{since } \varepsilon > 0)$$
$$||x_\varepsilon - x|| = \Big|\Big|\big[(1+\varepsilon)(x - x_0) + x_0\big] - x\Big|\Big| = \varepsilon \cdot r_0 \Rightarrow ||x_\varepsilon - x|| < r$$

(when ε is small enough)

Hence, $x_\varepsilon \in B(x,r) \backslash A, i.e. B(x,r) \not\subseteq A$.

3. if $||x - x_0|| < r_0$, then $B\big(x,\ r_0 - d(x, x_0)\big) \subseteq B(x_0, r_0)$ (we proved this earlier)

So, we have proven that $\mathring{A} = B(x_0, r_0)$. Again, this is not true for all metrics. For example, if d is the discrete metric and $r_0 = 1$, then $\{x \in X : d(x, x_0) \leq r_0\} = X$, and $\mathring{X} = X$, since $\forall x \in X, \quad B(x,1) \subseteq X$ (which is true $\forall\, r$). On the other hand, $B(x_0, r_0) = \{x_0\}$.

Remark 7.1

3. Case of the normed vector space $(X, ||.||)$

We will later prove this for all open sets, but we have $interior B(B(\mathring{x_0}, r_0) = B(x_0, r_0)$, from which it follows that $\partial B(x_0, r_0) = \overline{B(x_0, r_0)} \setminus B(\mathring{x_0}, r_0) = \{x \in X : ||x - x_0|| = r_0\}$ (with

$\partial B(x_0, r_0)$ being like the circumference).

Again, this is not true in general. If d is the discrete metric and $r = 1$, then $\partial B(x_0, r_0) = \overline{B(x_0, r_0)} \setminus \overset{\circ}{B}(x_0, r_0) = \{x_0\} \setminus \{x_0\} = \emptyset$.

Proposition 7.1

Let (X, d) be a metric space and $A \subseteq X$:

1. $\overline{X \setminus A} = X \setminus \mathring{A}$ and $\{X \overset{\circ}{\setminus} A\} = X \setminus \overline{A}$

2. $\partial A = \overline{A} \cap \overline{X \setminus A} = \partial(X \setminus A)$ (the boundary of a set is also the boundary of its complement.

Proof:

1.

$$\begin{aligned} x \in \overline{X \setminus A} &\iff \forall r > 0, \quad B(x, r) \cap (X \setminus A) = B(x, r) \setminus A \neq \emptyset \\ &\iff \text{not } \exists\, r > 0 \text{ such that } B(x, r) \setminus A = \emptyset \\ &\iff \text{not } \exists\, r > 0 \text{ such that } B(x, r) \subseteq A \\ &\iff \text{not } x \in \mathring{A} \iff x \in X \setminus \mathring{A} \end{aligned}$$

By applying the result to $X \setminus A$ instead of A, we obtain $\overline{X \setminus (X \setminus A)} = \overline{A} = X \setminus (X \overset{\circ}{\setminus} A)$, which gives $X \setminus \overline{A} = (X \overset{\circ}{\setminus} A)$

2. $\partial A = \overline{A} \setminus \mathring{A} = \overline{A} \cap (X \setminus \mathring{A}) = \overline{A} \cap \overline{X \setminus A}$ by property (1), which also implies $\partial A = \partial(X \setminus A)$ since we can exchange A and $X \setminus A$.

7.1 Properties of interior sets

Proposition 7.2

Let (X, d) be a metric space and $A \subseteq X$. Then $\mathring{A}$ is the **largest open set** in (X, d) that is contained in A in the sense that:

1. $\mathring{A} \subseteq A$

2. $\mathring{A}$ is open in (X, d)

3. $B \subseteq \mathring{A}$ for every open set B in (X, d), such that $B \subseteq A$

4. $A = \mathring{A}$ if and only if A is open in (X, d)

Remark 7.2

It also follows from (1), (2), and (3) that $\mathring{A}$ is the **unique** subset of X that satisfies those properties.

Proof. Indeed, if $\tilde{A}$ is another set satisfying (1), (2), (3), then it follows from (1) and (2) for $\mathring{A}$ and (3) for $\tilde{A}$ that $\mathring{A} \subseteq \tilde{A}$. Similarly, from (1) and (2) for $\tilde{A}$ and (3) for $\mathring{A}$, we get that $\tilde{A} \subseteq \mathring{A}$. Thus, forcibly, we have $\mathring{A} = \tilde{A}$ □

Proof of the properties from the proposition!

1. Let $x \in \mathring{A}$, i.e. $\exists\, r > 0$ such that $B(x, r) \subseteq A$. In particular, since $x \in B(x, r)$, it follows that $x \in A$. This proves that $\mathring{A} \subseteq A$.

2. Let $x \in \mathring{A}$, i.e. $\exists\, r > 0, \quad B(x, r) \subseteq A$. We want to show that $B(x, r) \subseteq \mathring{A}$. Let $y \in B(x, r)$. Since $B(x, r)$ is open, we obtain that $\exists\, s > 0$ such that $B(y, s) \subseteq B(x, r) \subseteq A$. This proves that $y \in \mathring{A}$, hence $B(x, r) \subseteq \mathring{A}$. Therefore, $\mathring{A}$ is open.

3. Let $B \subseteq A$ be such that B is open in (X, d). Let $x \in B$. Since B is open, $\exists\, r > 0, \quad B(x, r) \subseteq B$.
   Since $B \subseteq A$, it follows that $B(x, r) \subseteq A$, hence $x \in \mathring{A}$. This proves that $B \subseteq \mathring{A}$, and so $\mathring{A}$ is the largest open subset of A.

4. $A = \mathring{A}$, and so A is open because $\mathring{A}$ is open by property (2).
   Conversely, if A is open, then A is an open subset of itself, then since $A \subseteq A$, we can apply (3) with $B = A$, which gives $A \subseteq \mathring{A}$. By (1), we also have $\mathring{A} \subseteq A$; hence $A = \mathring{A}$. □

Lecture 6 - 01/23/2025

7.2 Properties of closure sets

Proposition 7.3

Let (X, d) be a metric space and $A \subseteq X$. Then, $\overline{A}$ is the smallest closed set in (X, d) that contains A, in the sense that:

1. $A \subseteq \overline{A}$
2. $\overline{A}$ is closed in (X, d)
3. $\overline{A} \subseteq B$ for every closed set B in (X, d) such that $A \subseteq B$
4. $A = \overline{A}$ if and only if A is closed in (X, d)

Remark 7.3

Again, (1), (2), and (3) imply that $\overline{A}$ is the unique set satisfying those properties.

Proof:

We apply the previous result to $X \backslash A$:

1. $(X \overset{\circ}{\backslash} A) = X \backslash \overline{A} \subseteq X \backslash A$, which gives $A \subseteq \overline{A}$
2. $(X \overset{\circ}{\backslash} A) = X \backslash \overline{A}$ is open, hence $\overline{A}$ is closed.
3. Let B be a closed set in (X, d) such that $A \subseteq B$. Then, $X \backslash B$ is open and $X \backslash B \subseteq X \backslash A$, which implies that $X \backslash B \subseteq (X \overset{\circ}{\backslash} A) = X \backslash \overline{A}$. Therefore, $\overline{A} \subseteq B$.
4. $A = \overline{A} \iff X \backslash A = X \backslash \overline{A} = (X \overset{\circ}{\backslash} A) \iff X \backslash A$ is open $\iff A$ is closed

Corollary 7.1

Let (X, d) br a metric space and $A \subseteq X$. Then ∂A is closed in (X, d).

Proof. $\partial A = \overline{A} \backslash \mathring{A}$ is closed since $\overline{A}$ is closed and $\mathring{A}$ is open.= $\square$

8 Convergence of sequences in metric spaces

Definition 8.1: Convergence

Let (X, d) be a metric space, let $x \in X$ and $(x_k)_{k\in\mathbb{N}}$ be a sequence of elements in X. We say that (x_k) converges to x in (X, d) *(or x is the **limit** of (x_k) in (X, d) and we denote by $\lim_{k\to\infty} x_k = x$ **or** $x_k \to x$ as $k \to \infty$)*

$$\text{if } \forall\, \varepsilon > 0, \;\; \exists\, k_\varepsilon \in \mathbb{N} \text{ such that } \;\forall k \geq k_\varepsilon, \;\; d(x, x_k) < \varepsilon \;\; \Big(i.e.\;\; x_k \in B(x, \varepsilon)\Big)$$

8.1 Properties of limits in metric spaces

Proposition 8.1

Let (X, d) be a metric space, $x \in X$ and $(x_k)_{k\in\mathbb{N}}$ be a sequence of elements in X.

1. $\lim_{k\to\infty} x_k = x$ in $(X, d) \iff \lim_{k\to\infty} d(x, x_k) = 0$ as a sequence in $(\mathbb{R}, |.|)$ (i.e. $\mathbb{R}$ equipped with the distance induced by the absolute value)

2. If $\lim_{k\to\infty} x_k \;\exists$ in (X, d) then it is **unique**

3. $\lim_{k\to\infty} x_k = x \iff$ for every subsequence $(x_{k_j})_{j\in\mathbb{N}}$ of $(x_k)_{k\in\mathbb{N}}$, we have $\lim_{k\to\infty} x_{k_j} = x$ *(recall that a subsequence of (x_k) is a sequence (x_{k_j}) such that $(k_j)_{j\in\mathbb{N}}$ is a sequence in $\mathbb{N}$ such that $k_{j+1} > k_j \;\; \forall\, j \in \mathbb{N}$)*

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Proof:

1. Recall from Analysis 1 that $\lim_{k\to\infty} d(x_k, x) = 0 \iff \forall\, \varepsilon > 0 \;\; \exists\, k_\varepsilon \in \mathbb{N} \;\; \forall k > k_\varepsilon \;\; |d(x_k, x) - 0| < \varepsilon$

2. Assume that $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} x_k = x'$. With the triangle inequality, we have $d(x, x') \leq d(x, x_k) + d(x_k, x')$. Since $\lim d(x, x_k) = \lim d(x', x_k) = 0$, by property (1), it follows that $0 \leq d(x, x') \leq \lim \Big(d(x, x_k) + d(x_k, x')\Big) = 0$. By the Squeeze theorem, this gives that $d(x, x') = 0$, i.e. $x = x'$.

3. $\lim_{k\to\infty} x_k = x \iff$ (by 1) $\lim_{k\to\infty} d(x_k, x) = 0 \iff$ (result from Analysis 1) $\lim_{k\to\infty} d(x_{k_j}, x) = 0$ for every subsequence $(x_{k_j}) \iff \lim_{k\to\infty} x_{k_j} = x$ for every subsequence (x_{k_j}).

$\square$

Example 8.1

$\lim_{k\to\infty} x_k = x$ in $(\mathbb{R}^n, ||.||_p) \iff \lim_{k\to\infty} (x_k)_i = x_i \;\; \forall\, i \in \{1, \ldots, n\}$, where $(x_k)_i$ and x_i are the i^{th} coordinate of x_k and x respectively.

Indeed, observe $0 \leq |(x_k - x)_i| \leq ||x_k - x||_p$, hence if $||x_k - x_p|| \to 0$, then $|(x_k - x)_i| \to 0 \;\; \forall$; $i \in \{1, \ldots, n\}$ as $k \to \infty$. Conversely, if $|(x_k - x)_i| \to 0 \;\; \forall\, i$, then $\max_{1\leq i\leq n} |(x_k - x)_i| \to 0$, with $p = \infty$ and $(\sum_{i=1}^n |(x_k - x)_i|^p)^{1/p} \to 0$; hence, $||x_k - x||_p \to 0$ as $k \to \infty$.

Example 8.2

Case of the discrete metric $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

$\lim_{k\to\infty} x_k = x$ in $(X, d) \iff \exists\, k_1 \in \mathbb{N}$ such that $\forall\, k \geq k_1, \;\; x_k = x$.

Indeed, if $\lim_{k\to\infty} x_k = x$, then $\exists\; k_1 \in \mathbb{N}$ such that $\forall\; k \geq k_1, \;\; d(x_k, x) < 1$ (, i.e. $d(x_k, x) = 0$, i.e. $x_k = x$).

Conversely, if $\exists\, k_1 \in \mathbb{N} \;\; \forall k \geq k_1, \;\; x_k = x$, then $\forall\, \varepsilon > 0 \;\; \forall k \geq k_1, \;\; d(x_k, x) = 0 < \varepsilon$.

Example 8.3

Case of $\left(C\left([a,b]\right), \; ||.||_\infty\right)$, where $a, b \in \mathbb{R}, \;\; a < b$ and $||f||_\infty = \max_{a\leq x\leq b}, |f(x)|$.

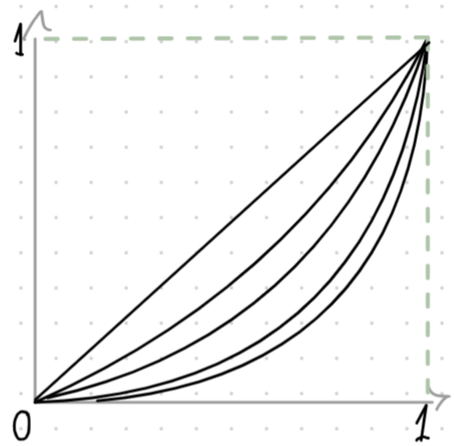
Then,

$$\begin{aligned} \lim_{k\to\infty} f_k = f \text{ in } \left(C\left([a,b]\right), ||.||_\infty\right) &\iff \forall\, \varepsilon > 0, \;\; \exists\, k_\varepsilon \in \mathbb{N}, \;\; \forall\, k \geq k_\varepsilon, \;\; ||f_k - f||_\infty < \varepsilon \\ &\iff \forall\, \varepsilon > 0, \;\; \exists\, k_\varepsilon \in \mathbb{N}, \;\; \forall k \geq k_\varepsilon, \;\; \max |f_k(x) - f(x)| < \varepsilon \\ &\iff \forall\, \varepsilon > 0, \;\; \exists\, k_\varepsilon \in \mathbb{N}, \;\; \forall\, k \geq k_\varepsilon, \\ &\qquad \forall\, x \in [a,b], \;\; |f_k(x) - f(x)| < \varepsilon \end{aligned}$$

We say that $(f_k)_{k\in\mathbb{N}}$ converges uniformly to f on $[a,b]$ or that f is the uniform limit of

$(f_k)_{k\in\mathbb{N}}$ on $[a,b]$.

From this last statement, let us give a little example. Say $a = 0, \quad b \in (0,1]$ and $f(x) = x^k \quad \forall\, x \in [a,b]$, with $k \in \mathbb{N}$. From these definition, we have $0 \le x^k \le b^k \quad \forall\, x \in [a,b]$; hence, $||f_k||_\infty \le b^k$.



If $b < 1$, then $b^k \to 0$, which would mean that $||f_k||_\infty \to 0$ as $k \to \infty$, i.e. $(f_k)_{k\in\mathbb{N}}$ converges uniformly to 0 in $[a,b]$.

If $b = 1$, then $||f_k||_\infty \ge |f_k(1)| = 1$; hence, $||f||_\infty \not\to 0$ as $k \to \infty$, i.e. $(f_k)_{k\in\mathbb{N}}$ does **not** converge uniformly to 0 in $[a,b]$.

Proposition 8.2

Let (X,d) be a metric space and $A \subseteq X$. Then, $x \in \overline{A}$ if and only if there exists a sequence $(x_k)_{k\in\mathbb{N}}$ of elements of A such that $\lim_{k\to\infty} x_k = x$ in (X,d)

Proof:

$(\Rightarrow)$: Assume that $x \in \overline{A}$. Then, $\forall\ k \in \mathbb{N}$, $A \cap B(x, \frac{1}{k}) \neq \emptyset$, i.e. $\exists\ x_k \in A \cap B(x, \frac{1}{k})$, which gives $x_k \in A$ and $d(x_k, x) < \frac{1}{k}$. Since $\lim_{k\to\infty} \frac{1}{k} = 0$, it follows that $\lim_{k\to\infty} d(x_k, x) = 0$, i.e. $\lim_{k\to\infty} x_k = x$.

$(\Leftarrow)$: Assume that $\exists\ (x_k)_{k\in\mathbb{N}}$ in A such that $\lim_{k\to\infty} x_k = x$. Then $\forall r > 0$, $\exists\ k_r \in \mathbb{N}\ \forall k \ge k_r, \quad d(x_k, x) < r$, i.e. $x_k \in B(x,r)$. Since $x_k \in A$ and $x_k \in B(x,r)$, we obtain $x_k \in A \cap B(x,r)$, which implies that $A \cap B(x,r) \neq \emptyset$. This proves that $x \in \overline{A}$. $\square$

9 Sequential Criterion for Closedness

Theorem 9.1: Sequential Criterion for Closedness

Let (X, d) be a metric space and $A \subseteq X$. Then, the following statements are equivalent:

1. A is closed in (X, d)

2. For every sequence $(x_k)_{k\in\mathbb{N}}$ of elements in A and every $x \in X$, if $\lim_{k\to\infty} x_k = x$ in (X, d), then $x \in A$. (in other words, for every subsequence in A, assume it converges to some point. If you can prove that the limit is in A for all subsequences, then A is closed!)

Proof:

$(i) \Rightarrow (ii)$: Assume that A is closed. Then $A = \overline{A}$. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in A and $x \in X$ be such that $\lim_{k\to\infty}\ x_k = x$. By the previous result, we then obtain $x \in \overline{A}$. Since $A = \overline{A}$, it follows that $x \in A$.

$(ii) \Rightarrow (i)$: Assume that (ii) is true. We want to show that $A = \overline{A}$. Since $A \subseteq \overline{A}$, if suffices to show that $\overline{A} \subseteq A$. Thus, let $x \in \overline{A}$. By the previous result, $\exists\ (x_k)_{k\in\mathbb{N}}$ in A such that $\lim_{k\to\infty} x_k = x$ in (X, d). BY (ii), it follows that $x \in A$. This proves that $\overline{A} \subseteq A$, and so, $A = \overline{A}$, i.e. A is closed in (X, d). $\square$

Lecture 8 - 01/30/2025

10 Cauchy Sequences

Definition 10.1: Cauchy sequences

Let (X, d) be a metric sequence and $(x_k)_{k\in\mathbb{N}}$ be a sequence of elements in X. We say that $(x_k)_k$ is a Cauchy sequence in (X, d) if $\forall\, \varepsilon > 0, \quad \exists\, k_\varepsilon \in \mathbb{N}$ such that $\forall\, k, k' \geq k_\varepsilon, \quad d(x_k, x_{k'}) < \varepsilon$.

Proposition 10.1

Every converging sequence in a metric space (X, d) is Cauchy in (X, d)

Proof:

Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in X such that $\lim_{k\to\infty} x_k = x$ for some $x \in X$. Then $\forall\, \varepsilon > 0,\ \ \exists\, k_\varepsilon \in \mathbb{N},\ \ \forall\, k \geq k_\varepsilon,\ \ d(x_k, x) < \varepsilon$. Then, $\forall\, k, k' \geq k_{\varepsilon/2},\ \ d(x_k, x_{k'}) \leq d(x_k, x) + d(x, x_{k'}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$; hence, $(x_k)_k$ is Cauchy. □

11 Complete Sets

Definition 11.1: Completeness

We say that a metric space (X, d) is complete if every Cauchy sequence in (X, d) converges to some limit in (X, d). A complete normed vector space is called a Banach space.
A complete inner product space is called a Hilbert space.

Example 11.1

1. $(\mathbb{R}, |.|)$ is complete (Banach) but $(\mathbb{Q}, |.|)$ is not complete. (Recall from Analysis 1 that $\mathbb{Q}$ is not compact in $\mathbb{R}$, so there is bound to be a sequence in $\mathbb{Q}$ that converges to an irrational number).

2. For every $n \in \mathbb{N}$ and $p \in [1, \infty]$, $(\mathbb{R}^n, ||.||_p)$ is Banach. Indeed, let $(x_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $(\mathbb{R}^n, ||.||_p)$

3. **Case of the discrete metric** $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

Proof of (2):

Observe that $\forall\, k, k' \in \mathbb{N},\ \ i \in \{1, \ldots n\},\ \ |(x_k - x_{k'})|_i \leq ||x_k - x_{k'}||_p$, where $|(x_k - x_{k'})|_i$ is the i^{th} coordinate of $(x_k - x_{k'})$. Hence, we obtain that $\big((x_k)_i\big)_{k\in\mathbb{N}}$ are also Cauchy in $(\mathbb{R}, |.|$. Thus, by completeness of $\mathbb{R}$, it follows that $\lim_{k\to\infty}(x_k)_i = a_i$ for some $a_i \in \mathbb{R}$, which implies that $\lim_{k\to\infty} x_k = a = (a_1, \ldots a_n)$ in $(\mathbb{R}^n, ||.||_p)$. □

Proof of (3):

Let $(x_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in (X, d). Then, $\exists\, k_1 \in \mathbb{N}\ \ \forall\, k, k' \geq k_1,\ \ d(x_k, x_{k'}) < 1$, i.e. $d(x_k, x_{k'}) = 0$, i.e. $x_k = x_{k'}$. Hence, $x_k = x_{k_1}\ \ \forall\, k \geq k_1$. It follows that $\lim_{k\to\infty} x_k = x_{k_1}$, which proves that (X, d) is complete in the case of the discrete metric. □

Proposition 11.1

Let (X, d) be a complete metric space, and $Y \subseteq X$. Then, (Y, d), as a metric subspace of (X, d), is complete if and only if Y is closed in (X, d).

Proof:

$(\Rightarrow)$: Assume that (Y, d) is complete. We will use the Sequential Criterion for Closedness. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in Y and let $x \in X$ be such that $\lim_{k\to\infty} x_k = x$. Then, $(x_k)_k$ is Cauchy in (X, d). Since $x_k \in Y$, it follows that $(x_k)_k$ is Cauchy in (Y, d). By completeness of (Y, d), it follows that $\lim_{k\to\infty} x_k = \tilde{x}$ for some $\tilde{x} \in Y$. Forcibly, by uniqueness of the limit in (X, d), it follows that $x = \tilde{x}$. Since $\tilde{x} \in Y$, we have $x \in Y$. This proves that Y is closed by the Sequential Criterion for Closedness.

$(\Leftarrow)$: Assume now that Y is closed in (X, d). Let $(x_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in (Y, d). Since $Y \subseteq X$, it follows that $(x_k)_k$ is also Cauchy in (X, d). By completeness of (X, d), it follows that $\lim_{k\to\infty} x_k = x$ for some $x \in X$. Since Y is closed, by the Sequential Criterion for Closedness, it follows $x \in Y$; hence, Y is complete. $\square$

12 Compact Sets and Metric Spaces

Definition 12.1: Compactness

Let (X, d) be a metric space and $A \subseteq X$. We say that A is compact or **sequentially compact** in (X, d) if for every sequence $(x_k)_{k\in\mathbb{N}}$ of elements in A, there exists $x \in A$ and a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_{k_j} = x$ in (X, d).

If this is true for $A = X$, then we say that (X, d) is a compact or sequentially **compact metric space**.

Example 12.1

$[a, b]$ are compact in $(\mathbb{R}, |.|)$ $\forall a, b \in \mathbb{R}$ such that $a < b$ (Analysis 1).

Proposition 12.1

Let (X, d) be a metric space and $A \subseteq X$:

1. If A is compact in (X, d), then A is closed in (X, d).

2. If A is compact in (X, d), then A is bounded in (X, d) in the sense that $\exists\, C > 0$ such that $\forall\, x, y \in A, \quad d(x, y) < C$.

3. If (X, d) is compact, then (X, d) is complete.

Remark 12.1

For every fixed $x_0 \in X$, it is easy to say that a set $A \subseteq X$ is bound if and only if there $\exists\, C > 0$, where C is a constant, such that $\forall\, x \in A, \quad d(x, x_0) < C$. (This definition is easier that using $\forall x, y \in X, \quad d(x, y) < C$ because instead of checking $\forall x$ and $\forall y$, you fix one, which is also more convenient).

Proof:

1. Assume that A is compact in (X, d). Let $(x_k)_{k \in \mathbb{N}}$ be a sequence in A and $x \in X$ such that $\lim_{k \to \infty} x_k = x$. Since A is compact, $\exists\, (x_{k_j})_{j \in \mathbb{N}}$ subsequence and $\tilde{x} \in A$ such that $\lim_{k \to \infty} x_{k_j} = \tilde{x}$. Moreover, since $\lim_{k \to \infty} x_k = x$, we have that $\lim_{k \to \infty} x_{k_j} = x$. By the uniqueness of limits, we have $x = \tilde{x}$. Now, since $\tilde{x} \in A$, it follows that $x \in A$. By the Sequence Criterion for Closedness, this proves that A is closed. $\square$

2. Indeed, if A is bounded, then fix $x_1 \in A$ (provided $A \neq \emptyset$, but the result is trivial otherwise) and write $\forall\, x \in A, \quad d(x, x_0) \leq d(x, x_1) + d(x_1, x_0)$. Since $d(x, x_1) < C$, then $d(x, x_0) < C_0$, where $C_0 = C + d(x_1, x_0)$., Conversely, if $\exists\, C > 0$ such that $\forall\, x \in A, \quad d(x, x_0) < C$, then $\forall\, x, y \in A$, $d(x, y) \leq d(x, x_0) + d(x_0, y) < 2C$. $\square$

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   There is also another way to prove this. We remark that for a fixed point x_0, A is bounded iff $\exists\, C > 0$ such that $\forall\, x \in A, \quad d(x, x_0) < C$. Assume that A is compact and, by contradiction, that A is not bounded. By the above remark, it follows that $\nexists\, C > 0 \quad \forall\, x \in A \quad d(x, x_0) < C$; hence $\forall\, k \in \mathbb{N} \quad \exists\, x_k \in A \quad d(x_k, x_0) \geq k$. By compactness of A, $\exists\, x \in A$ and a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_{k_j} = x$. Thus, it follows, from the triangle inequality, that $k \leq d(x_{k_j}, x_0) \leq d(x_{k_j}, x) + d(x, x_0)$. Since $\lim_{j\to\infty}\ d(x_{k_j}, x) = 0$ and $\lim_{j\to\infty}\ k_j = \infty$, we obtain a contradiction. Therefore, A is forcibly bounded. $\square$

3. Assume that (X, d) is compact. Let $(x_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in (X, d). By compactness of (X, d), $\exists\, x \in X$ and a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty}\ x_{k_j} = x$, i.e. $\forall\,\varepsilon > 0,\quad \exists\, j_\varepsilon \in \mathbb{N}$ such that $\forall\, j \geq j_\varepsilon,\quad d(x_{k_j}, x) < \varepsilon$. Since $(x_k)_{k\in\mathbb{N}}$ is Cauchy, we also have $\forall\,\varepsilon > 0,\quad \exists\, k_\varepsilon \in \mathbb{N}\quad \forall\, k, k' \geq k_\varepsilon,\quad d(x_k, x_{k'}) < \varepsilon$. Then, $\forall\,\varepsilon > 0$, letting $k' = k_j$, for j sufficiently large so that $j \geq j_{\varepsilon/2}$ and $k_j \geq k_{\varepsilon/2}$, we obtain that $\forall\, k \geq k_{\varepsilon/2},\quad d(x_k, x) \leq d(x_k, x_{k_j}) + d(x_{k_j}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$; hence $\lim_{k\to\infty}\ x_k = x$. Therefore, we have proved that every subsequence is Cauchy, and so (X, d) is complete. $\square$

Proposition 12.2

Let (X, d) be a compact metric space and $A \subseteq X$. Then A is compact in (X, d) if and only if A is closed in (X, d)

Proof:

The fact that A *compact* $\Rightarrow A$ *closed* follows from the previous result.

Assume that A is closed in (X, d). Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in A. By compactness of (X, d), $\exists\, x \in X$ and $(x_{k_j})_{j\in\mathbb{N}}$ a subsequence such that $\lim_{j\to\infty}\ x_{k_j} = x$. Since A is closed and $x_{k_j} \in A$, it follows that $x \in A$; hence, A is compact in (X, d). $\square$

13 Heine-Borel Theorem

Theorem 13.1: Heine-Borel Theorem

Let $n \in \mathbb{N},\quad p \in [1, \infty]$ and $A \subseteq \mathbb{R}^n$. Then, A is compact in $\mathbb{R}^n$ if and only if A is **closed and bounded** in $(\mathbb{R}^n, ||.||_p)$.

Remark 13.1

The Heine-Borel Theorem is not true in infinite dimensions, which will be proven later in the course.

Proof:

A *compact* $\Rightarrow A$ *closed* & *bounded* is always true and has already been proven.

Assume that A is closed and bounded in $(\mathbb{R}^n, ||.||_p)$. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in A. Recall that $|(x_k)_i| \leq ||x_k||_p \ \ \forall\ k \in \mathbb{N}, \ \ i \in \{1, \dots, n\}$ where $(x_k)_i$ is the i^{th} coordinate of x_k. Since $x_k \in A$ is bounded, we then obtain that $\big((x_k)_i\big)_{k\in\mathbb{N}}$ are bounded sequences in $(\mathbb{R}^n, ||.||_p)$. By repeatedly using the Bolzano-Weierstrass Theorem, it follows that there exists a subsequence of this subsequence such that the second coordinate also converges, etc. Finally, we obtain a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ such that all coordinates converges, i.e. $\lim_{j\to\infty}\ x_{k_j} = x$ for some $x \in \mathbb{R}^n$. Since A is closed and $x_{k_j} \in A$, we moreover have that $x \in A$. This proves that A is compact in $(\mathbb{R}^n, ||.||_p)$. $\square$

Proposition 13.1

Let (X, d) be a metric space and $A \subseteq X$. If A is compact in (X, d), then for every collection $\Big(U_i\Big)_{i\in I}$ of open sets in (X, d) such that $A \subseteq \bigcup_{i\in I} U_i$, there exists a finite sub-collection $\Big(U_{i_j}\Big)_{1\leq j\leq N}$, where $N \in \mathbb{N}$ and $i_1, \dots, i_n \in I$ such that

$$A \subseteq \bigcup_{j=1}^{N} U_{i_j}$$

Remark 13.2

It can be proven that the converse is also true. *(equivalent definition of compactness).*

Proof:

Assume that A is compact. Let $(U_i)_{i\in I}$ be open sets such that $A \subseteq \bigcup_{i\in I} U_i$.

Claim: $\exists\, r > 0$ such that $\forall\, x \in A, \ \ \exists\ i_x \in I$ such that $B(x, r) \subseteq U_{i_x}$. Assuming this claim as true, choose $x_1 \in A$ *(provided that $A \neq \emptyset$ but the result is trivial otherwise)* and, by induction, for each $j \in \mathbb{N}$, if $A \not\subseteq \bigcup_{i=1}^{j} B(x_i, r)$, then choose $x_{j+1} \in A \backslash \Big(\bigcup_{i=1}^{j} B(x_i, r)\Big)$. If the induction stops for some $j \in \mathbb{N}$, then $A \subseteq \bigcup_{i=1}^{j} B(x_i, r) \subseteq \bigcup_{i=1}^{j} U_{i_j}$ and the result is proven. If the induction does not stop, by compactness of A, $\exists\, x \in A$ and a subsequence $(x_{j_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\ x_{j_k} = x$. On the other hand, $x_{j_{k+1}} \notin \bigcup_{i=1}^{j_k} B(x_j, r)$ gives $d(x_{j_{k+1}}, x_j) \geq r \ \ \forall\, j \in \{1, \dots, j_k\}$; hence, $(x_{j_k})_{k\in\mathbb{N}}$ is not Cauchy by contradiction.

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Thus, to complete this proof in its entirety, all that is left is to prove the claim from above. Assume by contradiction that $B(x, r) \not\subseteq U_{i_x}$. Then, $\forall\ k \in \mathbb{N}, \ \ \exists\, x_k \in A$ such that $\forall\, i \in I, \ \ B(x_k, \frac{1}{k} \not\subseteq U_i$. By compactness of A, there exists a $x \in A$ and a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty}\ x_{k_j} = x$. Since $x \in A \subseteq \cup_{i\in I} U_i, \ \ \exists\, i_x \in I$ such that $x \in U_{i_x}$. Since $x \in U_{i_x}$ and U_{i_x} is open, $\exists\ r > 0$ such that $B(x, r_x) \subseteq U_{i_x}$.

On the other hand, since $B(x_k, \frac{i}{k}) \nsubseteq U_{i_x}$, $\exists\, y_k \in B(x_k, \frac{1}{k}) \backslash U_{i_x}$. Then, $d(x, y_{k_j}) \leq d(x, x_{k_j}) + d(x_{k_j}, y_{k_j})$. Consider now that $d(x_{k_j}, y_{k_j}) < \frac{1}{k_j}$ (since $y_{k_j} \in B(x_k, \frac{i}{k})$ and $d(x, x_{k_j}) \to 0$ since $x_{k_j} \to x$. This means that $d(x, y_{k_j}) \to 0$, i.e. $y_{k_j} \to x$ as $j \to \infty$. However, $y_{k_j} \notin U_{i_x}$ and $B(x, r_x) \subseteq U_{i_x}$; hence $y_{k_j} \notin B(x, r_x)$. This brings us to a contradiction between $d(x, \frac{1}{k_j}) \to 0$ and $d(x, y_{k_j}) \geq r_x$. $\square$

Corollary 13.1

Let (X, d) be a metric space and $A \subseteq X$. If A is compact, then for every $r > 0$, there exists a finite number of elements $x_1, \ldots, x_{N_r} \in A$, where $N_r \in \mathbb{N}$ such that

$$A \subseteq \bigcup_{i=1}^{N_r} B(x_i, r)$$

Proof. This is a direct application of the proposition to the open cover $A \subseteq \bigcup_{x \in A} B(x, r)$. $\square$

Corollary 13.2

Let $(X, ||.||)$ be a normed vector space of infinite dimension. Then, $B(0, 1)$ is not compact. (hence, the Heine-Bornel Theorem is not true in infinite dimensions).

Proof. Assume by contradiction that $\overline{B(0,1)}$ is compact. By corollary 13.1, it follows that $\exists\, N \in \mathbb{N}$ and $\exists\, x_1, \ldots, x_N \in \overline{B(0,1)}$ such that $\overline{B(0,1)} \subseteq \cup_{i=1}^{N} B(x_i, \frac{1}{2})$ (see Figure 3 for visualization). Since X has infinite dimensions, $\exists\ x_0 \notin Span(x_1, \ldots, x_N)$. By renormalizing (i.e. considering λx, for some $\lambda > 0$, instead of x_0), we may choose x_0 such that $x_0 \in B(0, \frac{1}{2})$.

By induction, we prove the following: $\forall\, k \in \mathbb{N}$, $\exists\, y_k, z_k \in X$ such that $x_0 = y_k + z_k$, $||y_k|| < 2^{-k}$ and $z_k \in Span(x_1, \ldots, x_N)$. For $k = 1$, it is trivial by taking $y_k = x_0$ and $z_k = 0$. Now, let us assume that it is true $\forall\, k$. Observe that $||2^k y_k|| = 2^k ||y_k|| < 1$, i.e. $2^k y_k \in B(0, 1)$. Since $B(0, 1) \subseteq \bigcup_{i=1}^{N} B(x_i, \frac{1}{2})$, it follows that $\exists\, i_k \in \{1, \ldots, N\}$ such that $2^k y_k \in B(x_{i_k}, \frac{1}{2})$, i.e. $||2^k y_k - x_{i_k}|| < \frac{1}{2}$, which gives $||y_k - 2^{-k} y_{i_k}|| < 2^{-k-1}$. Now write:

$$x_0 = \overbrace{(y_k - 2^{-k} x_{i_k})}^{y_{k+1}} + \overbrace{(2^{-k} x_{i_k} + z_k)}^{z_{k+1}} \Rightarrow ||y_{k+1}|| < 2^{-k-1} \text{ and } z_{k+1} \in Span(x_1, \ldots, x_N), \text{ since } z_k \in Span$$

This proves that $\forall\, k \in \mathbb{N}$, $\exists\, y_k, z_k \in X$ such that $x_0 = y_k + z_k$, $||y_k|| < 2^{-k}$ and $x_k \in Span(x_1, \ldots, x_N)$ holds also for $k + 1$.

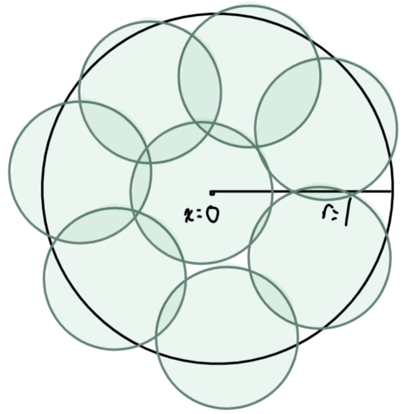

Figure 3: Representation of $\cup B(x_i, 1/2)$

Since $||y_k|| < 2^{-k}$, we obtain that $||x_0 - z_k|| \to 0$, i.e. $z_k \to x_0$ in $(X, ||.||)$. Since $z_k \in Span(x_1, \dots, x_N)$, it follows that $x_0 \in Span(x_1, \dots, x_N)$. However, $x_0 \notin Span$. It remains to show that $Span(x_1, \dots, x_N)$ is closed in $(X, ||.||)$ to obtain a contradiction. We will prove later that any subspace of finite dimensions are always closed. $\square$

14 Convergence of functions

Definition 14.1: Limits of functions

Let (X, d_x) and (Y, d_y) be metric spaces with $x_o \in X, \quad y_0 \in Y$.

1. **Regular Limits:** For $f : A \to Y$, we say that $f(x)$ converges to y_0 as $x \to x_0$ and we denote $\lim_{x \to x_0} f(x) = y_0$ if (see Figure) $\forall\, \varepsilon > 0, \quad \exists\, \delta_\varepsilon > 0$ such that $\forall\, x \in A$,

$$0 < d_X(x, x_0) < \delta_\varepsilon \Rightarrow d_Y\Big(f(x), y_0\Big) < \varepsilon, \quad i.e. \quad f\Big(A \cap B_X(x_0, \delta_\varepsilon) \backslash \{x_0\}\Big) \subseteq B_Y(y_0, \varepsilon)$$

2. **Limits of infinite sets:** When $(Y, d_Y) = (\mathbb{R}, |.|)$, for $f : A \to \mathbb{R}$, we say that $f(x)$ tends to $\binom{+\infty}{-\infty}$ and we denote $\lim_{x \to x_0} f(x) = \binom{+\infty}{-\infty}$ if $\forall\, c \in \mathbb{R}, \quad \exists\ \delta_c > 0$ such that $\forall\, x \in A$

$$0 < d_X(x, x_0) < \delta_c \Rightarrow f(x) \begin{pmatrix} > c \\ < c \end{pmatrix}, \quad i.e.\ f\Big(A \cap B_X(x_0, \delta_c) \backslash \{x_0\}\Big) \subseteq \begin{pmatrix} (c, +\infty) \\ (-\infty, c) \end{pmatrix}$$

3. **Limits in infinite metric spaces:** When $(X, d_X) = (\mathbb{R}, |.|)$ and $A \supseteq \binom{(c, +\infty)}{(-\infty, c)}$ for some $c \in \mathbb{R}$ for $f : A \to Y$, we say that $f(x)$ converges to y as $x \to \binom{+\infty}{-\infty}$ and we denote

$\lim_{x\to\binom{+\infty}{-\infty}} f(x) = y_0$ if $\forall\, \varepsilon > 0,\quad \exists\, c_\varepsilon \in \mathbb{R}$ such that $\forall\, x \in A$

$$x\begin{pmatrix} > c_\varepsilon \\ < c_\varepsilon \end{pmatrix} \Rightarrow d_Y\big(f(x), y_0\big) < \varepsilon, \quad i.e.\ f\left(A \cap \begin{pmatrix} (c_\varepsilon, \infty) \\ (-\infty, c_\varepsilon) \end{pmatrix}\right) \subseteq B_Y(y_0, \varepsilon)$$

14.1 Continuity in Metric Spaces

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Definition 14.2: Continuity of functions

Let (X, d_X) and (Y, d_Y) be metric spaces, $A \subseteq X,\ \ x_0 \in A$ and $f : A \to Y$. We say that f is continuous at x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$, i.e. $\forall\ \varepsilon > 0,\ \ \exists\, \delta_\varepsilon > 0$ such that $\forall\, x \in A,\ \ d_X(x, x_0) < \delta_\varepsilon \Rightarrow d_Y\big(f(x), f(x_0)\big) < \varepsilon$. We say that f is continuous on $B \subseteq A$ if f is continuous at every $x_0 \in B$ (or simply continuous when $B = A$)

Example 14.1

1. The identity function $d_X : X \to X,\ \ x \mapsto x$ is continuous. *Proof* is trivial: $\forall\, \varepsilon > 0,\ \ d_X(x, x_0) < \delta_\varepsilon = \varepsilon \Rightarrow d_X\big(Id_X(x) Id_X(x_0)\big) = d_X(x, x_0) < \varepsilon$.

2. For $y_0 \in X$ fixed, the function $d(., y_0) : X \to X,\ \ x \mapsto d(x, y_0)$ is continuous. Indeed, observe that $\forall\, x, x_0 \in X,\ \ d(x, y_0) \le d(x, x_0) + d(x_0, y)$, which gives $d(x, y_o) - d(x_0, y) \le d(x, x_0)$ and similarly $d(x_0, y_0) - d(x, y_0) \le d(x, x_0)$. Hence, $\Big|d(x, y_0) - d(x_0, y_0)\Big| \le d(x, x_0)$. It follows that $\forall\, \varepsilon > 0,\ \ d(x, x_0) < \delta_\varepsilon = \varepsilon \quad \Rightarrow \big|d(x, y_0) - d(x_0, y_0)\big| < \varepsilon$.

   In particular, this shows that in a normed vector space $(X, ||.||)$, the norm $||.|| : X \to \mathbb{R}$ is **continuous** since in this case, $||X|| = d(x, 0)$.

3. In an **inner product space** $\big(X, \langle ., .\rangle\big)$, for $y_0 \in X$ fixed, the function $X \to \mathbb{R},\ x \mapsto \langle x, y_0\rangle$, is continuous. Indeed, $\forall\ x, x_0 \in X$

$$\big|\langle x, y_0\rangle - \langle x_0, y_0\rangle\big| = \big|\langle x - x_0, y_0\rangle\big| \le ||x - x_0|| \cdot ||y_0|| \quad \text{by Cauchy-Schwartz}$$

Then, $\forall\, \varepsilon > 0$

$$||x - x_0|| < \delta_\varepsilon = \begin{cases} \varepsilon \Big/ ||y_0|| & \text{if } y_0 \neq 0 \\ any \quad number & \text{if } y_0 = 0 \end{cases} \quad \Rightarrow \quad \big|\langle x, y_0\rangle - \langle x_0, y_0\rangle\big| \leq \frac{\varepsilon}{||y_0||} \cdot ||y_0|| = \varepsilon$$

4. For each $n_1, n_2 \in \mathbb{N}, \quad p_1, p_2 \in [1, \infty], \quad A \subseteq \mathbb{R}^{n_1}, \quad x_0 \in A$ and $f : A \to \mathbb{R}^{n_2}, \quad f : x \mapsto \Big(f_1(x), \ldots, f_2(x)\Big)$. Then, f is continuous at x_0 if and only if f_i is continuous at $x_0 \quad \forall\, i \in \{1, \ldots, n_2\}$???? i dont fucking know girlie pop

Proposition 14.1

Let (X, d_X) and (Y, d_Y) be metric spaces, with $A \subseteq X, \quad x_0 \in \overline{A \backslash \{x_0\}}$[a] and $f : A \to Y$. If $\lim_{x \to x_o} f(x)$ exists, then it is unique

[a]Exclude x_0 to avoid the case where x_0 is isolated in the sense that $\exists\, r > 0$ such that $B(x_0, r) \cap A = \{x_0\}$

Proof. By contradiction, assume that $\exists\, y_0 \neq \tilde{y}_0$ such that $\lim_{x \to x_0} f(x) = y_0$ and $\lim_{x \to x_0} f(x) = \tilde{y}_0$, i.e.

- $\forall \varepsilon >, \quad \exists\, \delta_\varepsilon > 0 \quad \forall\, x \in A, \quad 0 < d_X(x, x_0) < \delta_\varepsilon \Rightarrow d_Y(f(x), y_0) < \varepsilon$

- $\forall\, \varepsilon > 0, \quad \exists\, \tilde{\delta}_\varepsilon > 0, \quad \forall\, x \in A \quad 0 < d_X(x, x_0) < \tilde{\delta}_\varepsilon \Rightarrow d_Y(f(x), \tilde{y}_0) < \varepsilon$

Define $\delta'_\varepsilon = \min(\delta_{\varepsilon/2}, \tilde{\delta}_{\varepsilon/2})$. Since $x_0 \in \overline{A \backslash \{x_0\}}, \quad \exists\, x_\varepsilon \in B(x_0, \delta'_\varepsilon) \cap \Big(A \backslash \{x_0\}\Big)$, i.e. $x_\varepsilon \in A$ and $0 < d_X(x_\varepsilon, x_0) < \delta'_\varepsilon$, which implies that $d_Y(f(x), y_0) < \frac{\varepsilon}{2}$ and $d_Y(f(x), \tilde{y}_0) < \frac{\varepsilon}{2}$. It follows that $d_Y(y_0, \tilde{y}_0) \leq d_Y\big(y_0, f(x)\big) + d_Y\big(f(x_\varepsilon), \tilde{y}_0\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Since this is true $\forall\, \varepsilon > 0$, it follows that $d_Y(y_0, \tilde{y}_0) = 0$, which is a contradiction. Thus, $y_0 = \tilde{y}_0$. □

Lecture 12 - 02/13/2025

Proposition 14.2

Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces, $x_0 \in X, \quad f : X \to Y$ and $g : Y \to Z$. If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0. More generally, if f is continuous on $A \subseteq X$ and g is continuous on $f(A)$, then $g \circ f$ is continuous on A.

Proof. If f is continuous at x_0, then $\forall\ \varepsilon > 0, \quad \exists\ \delta_\varepsilon > 0$ such that $\forall\, x \in X, \quad d_X(x, x_0) < \delta_\varepsilon \Rightarrow d_Y\big(f(x), f(x_0)\big) < \varepsilon$. If g is continuous at $f(x_0)$, then $\forall \varepsilon > 0, \quad \exists\ \delta'_\varepsilon > 0$ s.t. $\forall\, y \in Y, \quad d_Y\big(y, f(x_0)\big) < \delta'_\varepsilon \Rightarrow d_Z\big(g(y), g(f(x_0))\big)$.

$\forall \varepsilon > 0$, letting $\delta''_\varepsilon = \delta_{\delta'_\varepsilon}$, it follows that $\forall x \in X$, $d_X(x, x_0) < \delta''_\varepsilon \Rightarrow d_Y\big(f(x), f(x_0)\big) < \delta'_\varepsilon \Rightarrow d_Z\Big(g\big(f(x)\big), g\big(f(x_0)\big)\Big) < \varepsilon$. Therefore, $g \circ f$ is continuous at x_0[1] $\square$

Proposition 14.3

Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$. Then, the following statements are **equivalent**:

- f is continuous from (X, d_X) to (Y, d_Y)
- For every open set O in (Y, d_Y), $f^{-1}(O)$ is open in (X, d_X)
- For every closed set C in (Y, d_Y), $f^{-1}(C)$ is closed in (X, d_X)

Remark 14.1

The inverse is not true. There exists open sets O in (X, d_X) such that $f(O)$ is not open in (Y, d_Y), even if f is continuous. The same is true for closed sets. Thus, the proposition is only in regard from the function to the preimage, and doesn't work the other way around.

Example 14.2

Let $f : \mathbb{R} \to \mathbb{R}$ with $f : x \mapsto \frac{1}{1+x^2}$. $\mathbb{R}$ is open and closed, where as $f(\mathbb{R}) = (0, 1]$ is not open, nor closed.

Proof. **1) $\Rightarrow$ 2):** Assume that f is continuous. Let O be an open set in (Y, d_Y). Let $x_0 \in f^{-1}(O)$, i.e. $f(x_0) \in O$/ We want to show that $\exists\, r > 0$ such that $B_Y\big(f(x_0), \varepsilon\big) \subseteq O$. Since f is continuous, it follows that $\exists\, \delta_\varepsilon > 0$ such that $f\big(B_X(x_0, \delta_\varepsilon)\big) \subseteq B_Y\big(f(x_0), \varepsilon\big) \subseteq O$, which gives $V_X(x_0, \delta_\varepsilon) \subseteq f^{-1}(O)$. This proves that $f^{-1}(O)$ is open in (X, d_X)

2) $\Rightarrow$ 1): Assume that 2) is true. Let $x_0 \in X$ and $\varepsilon > 0$. Since $B\big(f(x_0), \varepsilon\big)$ is open in (Y, d_Y), it follows from 2) that $f^{-1}\Big(B_Y\big(f(x_0), \varepsilon\big)\Big)$ is open in (X, d_X). Since $f(x_0) \in B_Y\big(f(x_0), \varepsilon\big)$, i.e. $x_0 \in f^{-1}\Big(B_Y\big(f(x_0), \varepsilon\big)\Big)$ and $f^{-1}\Big(B_Y\big(f(x_0), \varepsilon\big)\Big)$ is open in (X, d_X), we then obtain that $\exists\, \delta_\varepsilon > 0$

[1] See how when proving that a composed function is continuous, we kind of formed a "composed" delta-epsilon.

such that $B_X(x_0, \delta_\varepsilon) \subseteq f^{-1}\Big(B_Y\big(f(x_0, \varepsilon\big)\Big)$, which gives $f\Big(B_x(x_0, \delta_\varepsilon)\Big) \subseteq B_Y\big(f(x_0), \varepsilon\big)$. This proves that f is continuous.

2) $\Rightarrow$ 3): Assume that 2) is true. Let C be a closed set in (Y, d_Y). Then, $Y \backslash C$ is open; hence, by 2), $f^{-1}(Y \backslash C)$ is open. Since $f^{-1}(Y \backslash C) = X \backslash f^{-1}(C)$, it follows that $X \backslash f^{-1}(C)$ is open, i.e. $f^{-1}(C)$ is closed in (X, d_X).

3) $\Rightarrow$ 2): Direct proof using the same logic as above. $\square$

14.2 Sequence Criterion for Continuity

Theorem 14.1: Sequence Criterion for Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces, $x_0 \in X$ and $f : X \to Y$. Then, the following statements are equivalent:

- f is continuous at x_0
- For every sequence $(x_k)_{k \in \mathbb{N}}$ of elements in X, if $\lim_{k \to \infty} x_k = x_0$, then $\lim_{k \to \infty} f(x_k) = f(x_0)$.

Proof. **1) $\Rightarrow$ 2):** Assume that f is continuous at x_0. Let $(x_k)_{k \in \mathbb{N}}$ be a sequence in X such that $\lim_{k \to \infty} x_k = x_0$.
Since f is continuous at x_0, $\forall \varepsilon > 0$, $\exists\, \delta_\varepsilon > 0$ s,t, $\forall\, x \in X$, $d_X(x, x_0) < \delta_\varepsilon \Rightarrow d_Y\big(f(x), f(x_0)\big) < \varepsilon$.
Since $\lim_{k \to \infty} x_k = x_0$, $\exists\, k_{\delta_\varepsilon} \in \mathbb{N}$ s.t. $\forall\, k \geq k_{\delta_\varepsilon}$, $d_X(x, x_0) < \delta_\varepsilon$, which implies that $d_Y\big(f(x_k), f(x_0)\big) < \varepsilon$. This implies that $\lim_{k \to \infty} f(x_k) = f(x_0)$.

2) $\Rightarrow$ 1): Assume that 2) is true, and, by contradiction, that f is not continuous at x_0, i.e. $\exists\, \varepsilon > 0$, s.t. $\forall k \in \mathbb{N}$, $\exists\, x_k \in X$ s.t. $d_X(x_0, x_k) < \frac{1}{k}$, and $d_Y\big(f(x_0), f(x_k)\big) \geq \varepsilon$. Since $d_X(x_0, x_k) < \frac{1}{k} \to 0$ as $k \to \infty$, we obtain $\lim_{k \to \infty} x_k = x_0$. It follows from 2) that $\lim_{k \to \infty} f(x_k) = f(x_0)$, i.e. $\lim_{k \to \infty} d_Y\big(f(x_0), f(x_k)\big) = 0$, but this is in contradiction with $d_Y\big(f(x_0), f(x_k)\big) \geq \varepsilon$. Therefore, f is continuous at x_0. $\square$

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Proposition 14.4

Let (X, d_X) and (Y, d_Y) be metric spaces, $A \subseteq X$ and $f : X \to Y$. If A is compact in (X, d_X) and f is continuous on X, then $f(A)$ is compact in (Y, d_Y).

Example 14.3

1. There exists compact sets A in (Y, d_Y) such that $f^{-1}(A)$ is not compact for some continuous $f : X \to Y$. Example: $f(x) = \frac{1}{1+x^2}$. Say $A = [0, 1]$, $\quad f^{-1}([0, 1]) = \mathbb{R}$, which is not compact, while $[0, 1]$ is compact.

2. There exists non-continuous functions f such that $f(A)$ is compact for all compact $A \subseteq X$. Example: $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, such that $f(A) \subseteq \{0, 1\} \quad \forall A \subseteq \mathbb{R}$ and every subset of $\{0, 1\}$ is compact.

Proof. Assume that f is continuous and A is compact. Let $(\frac{1}{k})_{k\in\mathbb{N}}$ be a sequence in $f(A)$. Then, $\exists\, x_k \in X$ such that $f(x_k) = y_k$. Since A is compact, $\exists\, x \in A$ and a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} f(x_{k_j}) = f(x)$, i.e. $\lim_{j\to\infty} y_{k_j} = f(x)$. Moreover, $f(x) \in f(A)$ since $x \in A$. This proves that $f(A)$ is compact. □

Proposition 14.5

Let (X, d) be a metric space, $A \subseteq X, \quad A \neq \emptyset$ and $f : X \to \mathbb{R}$. If A is compact in (X, d) and f is continuous on X, then $\exists\, x_{min}, x_{max} \in A$ such that

$$f(x_{\min}) = \min_{x\in A} f(x) \quad \text{and} \quad f(x_{\max}) = \max_{x\in A} f(x)$$

Proof. By the previous result, we have that $f(A)$ is compact. In particular, $f(A)$ is bounded. Moreover, $A \neq \emptyset$, hence $f(A) \neq \emptyset$. Therefore, $\inf f(A) > -\infty$ and $\sup f(A) < \infty$. It follows that there exists sequence $(m_k)_{k\in\mathbb{N}}$ and $(M_k)_{k\in\mathbb{N}}$ in $f(A)$ such that $\lim_{k\to\infty} m_k = \inf f(A)$ and $\lim_{k\to\infty} M_k = \sup f(A)$.

Since $f(A)$ is compact, we obtain that $f(A)$ is closed, hence, since $m_k, M_k \in A$, the sequence criterion for closedness gives that $\inf f(A) \in f(A)$ and $\sup f(A) \in f(A)$, i.e. $\exists\, x_{\min}, x_{\max} \in A$ such that $f(x_{\min}) = \inf f(A)$ $(= \min f(A)$ since it is attained), and $f(x_{\max}) = \sup f(A)$ $(= \max f(A)$ since it is attained). □

Remark 14.2

We can now define $||f||_\infty = \max_{x\in A}|f(x)|$ for ever continuous function $f : X \to \mathbb{R}$, when A is compact and non-empty in (X, d). We denote $C(A)$ the set of continuous functions on A. Like in $Q2$ of Assignment 3, we obtain that $\big(C(A), ||.||_\infty\big)$ is a Banach space.

Definition 14.3: Uniform Continuity

Let (X, d_X) and (Y, d_Y) be two metric spaces, $A \subseteq X$ and $f : X \to Y$. We say that f is uniformly continuous on A if $\forall\ \delta_\varepsilon > 0$ such that $\forall\ x, y \in A,\ \ d_X(x, y) < \delta_\varepsilon \Rightarrow d_Y\big(f(x), f(y)\big) < \varepsilon$

Uniformly continuous $\Rightarrow$ Continuous

Definition 14.4: Lipschitz Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces, $A \subseteq X$ and $f : X \to Y$. We say that f is Lipschitz continuous on A if $\exists\, C \in (0, \infty)$ such that $\forall\, x, y \in A,\ \ d_Y\big(f(x), f(y)\big) \leq C d_X(x, y)$

Lipschitz continuous $\Rightarrow$ Uniform continuous

Example 14.4

1. For each $y_0 \in X$, the function $x \mapsto d(x, y_0)$ is Lipschitz continuous. Indeed, we proved that $\forall\ x, x' \in X,\ \ |d(x, y_0) - d(x', y_0)| \leq d(x, x')$.

2. Let I be an interval. in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be differentiable at every point in I and such that $\sup_{x\in I}|f'(x)| < \infty$. Indeed, by the Mean Value Theorem, $\forall\, x, y \in I,\ \ \exists\ z \in I$ between x and y such that $f(x) - f(y) = f'(z)(x - y)$, which gives $|f(x) - f(y)| \leq \sup_{z\in I}|f'(z)|\,|x - y|$ which is smaller than ∞ by assumption.

Proposition 14.6

Let (X, d_X) and (Y, d_Y) be metric spaces, $A \subseteq X$ and $f : X \to Y$. If A is compact in (X, d_X) and f is continuous on A, then f is uniformly continuous on A.

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Proposition 14.7

Let (X, d), (Y, d_Y) be metric spaces, $A \subseteq X$ and $f : X \to Y$.

If A is compact in (X, d) and f is continuous on A, then f is uniformly continuous on A

Proof. Assuming by contradiction that f is not uniformly continuous, we obtained the existence of $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ such that $d_X(x_k, y_k) < \frac{1}{k}$ and $d_Y\big(f(x_k), f(y_k)\big) \geq \varepsilon$ for some fixed $\varepsilon > 0$.

By using compactness of A, we then obtained subsequences $(x_{k_j})_{j\in\mathbb{N}}$ and $(y_{k_{j_i}})_{i\in\mathbb{N}}$ such that $\lim_{j\to\infty}\ x_{k_j} = x$ and $\lim_{i\to\infty}\ y_{k_{j_i}} = y$ for some $x, y \in A$. By using $d(x_k, y_k) < \frac{1}{k}$, we then obtained $x = y$.

By continuity of f, since $\lim_{j\to\infty}\ x_{k_j} = x$ and $\lim_{i\to\infty}\ y_{k_{j_i}} = y = x$, we have that $\lim_{j\to\infty}\ f(x_{k_j}) = f(x)$ and $\lim_{i\to\infty}\ f(y_{k_{j_i}}) = f(y) = f(x)$.

It follows that

$$d_Y\big(f(x_{k_j}), f(y_{k_{j_i}})\big) \leq d_Y\big(f(x_{k_j}), f(x)\big) + d)Y\big(f(y), f(y_{k_{j_i}})\big)$$

and, since $\lim_{j\to\infty}\ f(x_{k_j}) = f(x)$ and $\lim_{i\to\infty}\ f(y_{k_{j_i}}) = f(y) = f(x)$, then both distances approach 0. Hence, $\lim_{i\to\infty}\ d_Y\big(f(x_{k_j}), f(y_{k_{j_i}}) = 0\big)$, but this contradicts that $d_Y\big(f(x_{k_j}), f(y_{k_{j_i}})\big) < \varepsilon$. $\square$

Definition 14.5: L

et (X, d) be a metric space, $A \subseteq X$ and $f : X \to X$.

We say that f is **continuous** on A if

$$\exists\, C \in (0, 1) \text{ such that } \forall\, x, y \in A, \quad d_X\big(f(x), f(y)\big) \leq C d_X(x, y)$$

Remark 14.3

1. Contraction $\Rightarrow$ Lipschitz continuous

2. $C < 1$: $f(x)$ is "closer" to $f(y)$ then x is to y

15 Banach Fixed-Point Theorem

Theorem 15.1: Fixed-Point

Let (X, d) be a non-empty complete metric space and $f : X \to X$ be a contraction on X. Then, f has a unique fixed point in X, namely there exists a unique $x_0 \in X$ such that $f(x_0) = x_0$.

Example 15.1

1. Let $I \subseteq X$ be a closed interval in $\mathbb{R}$ (so that $(I, |.|)$ is complete), and $f : I \to I$ be differentiable on I, such that $\sup_{x \in I} |f'(x)| < 1$. Then, f is a contraction on I; hence, it has a unique fixed point in I.

2. Let $X = [0, \infty)^n = \{(x_1, \ldots, x_n)\} \geq 0 \ \forall i \in \{1, \ldots, n\}$ and $f : X \to X$ with $x \mapsto \lambda x + a$ for some $\lambda > 0$ and $a \in X$. Then, X is complete (as closed subset of $(\mathbb{R}^n, ||.||_2)$). Moreover, f is a contraction on X if and only if $\lambda < 1$. Indeed, $\forall x, y \in X, \quad ||f(x) - f(y)|| = ||\lambda(x, y)|| = \lambda||x - y||$.

   Note that if $\lambda \geq 1$ and $a \neq 0$, then f has no fixed point. Indeed, if $f(x) = x$, then $\lambda x + a = x$ gives $(\lambda - 1)x + a = 0$. Since $\lambda - 1 > 0, \quad x \geq 0$ and $x \geq 0 \quad \forall\, i \in \{1, \ldots, n\}$, it follows that $(\lambda - 1)x = a = 0 \quad \forall i \in \{1, \ldots, n\}$ which contradicts $a \neq 0$.
   Note also that if $X = (0, \infty)^n$ and $f : X \to X$, with $x \mapsto \lambda x$, then f is a contraction on X when $\lambda \in (0, 1)$, but $(X, ||.||_2)$ is not complete (since $(0, \infty)^n$ is not closed in $(\mathbb{R}^n, ||.||_2)$) and f has no fixed point in X.

Proof. **Existence:** Since X is not empty, we can choose $x_1 \in X$. Then, define $(x_k)_{k \in \mathbb{N}}$ by $x_{k+1} = f(x_k)$. Since f is a contraction, $\exists\, C \in (0, 1)$ such that $\forall\, x, y \in X, \quad d_X\big(f(x), f(y)\big) \leq C d_X(x, y)$, hence $\forall\, k \in \mathbb{N}, \quad d\big(x_{k+2}, x_{k+1}\big) = d\Big(f(x_{k+1}), f(x_k)\Big) \leq C d_X(x_{k+1}, x_k)$ $\square$

2 Infinite Series

2 Convergence of Infinite Series

Lecture 15 - 02/25/2025

Definition 2.1: Convergence of Infinite Series

Let $(X, ||.||)$ be a normed vector space, $k_0 \in \mathbb{Z}$ and $(a_k)_{k \geq k_0}$ be a sequence of elements in X. For each $n \geq k_0$, we call $s_n = \sum_{k=k_0}^{n} a_k$ the **partial sum of order** n of the series $\sum_{k=k_0}^{\infty} a_k$. If $\lim_{n\to\infty} s_n = s$ in $(X, ||.||)$ for some $s \in X$, then we denote $s = \sum_{k=k_0}^{\infty} a_k$ and we say that the series $\sum_{k=k_0}^{\infty} a_k$ **converges** to s and that s is the **sum** of the series $\sum_{k=k_0}^{\infty} a_k$.

If $(s_n)_{n\in\mathbb{N}}$ does not converge to any limit in $(X, ||.||)$, then we say that the series $\sum_{k=k_0}^{\infty} a_k$ **diverges**.

Example 2.1

1. Geometric series: $a_k = r^k$, $\sum_{k=0}^{n} r^k = \begin{cases} \frac{1-r^{n+1}}{1-r} & \text{if } r \neq 1 \\ n+1 \end{cases}$. $\sum_{k=0}^{\infty} r^k$ converges if and only if $|r| < 1$. If $|r| < 1$, then $\sum_{k=0}^{\infty} r^k - \frac{1}{1-r}$

2. Let $M(n, \mathbb{R})$ be the vector space of $n \times n$ real matrices equipped with the norm

$$||A||_{2,2} = \sup_{||x||_2=1} ||Ax||_2$$

(= $\max_{||x||=1} ||Ax||$ since $x \mapsto ||Ax||_{2,2}$ is continuous [see Assignment 4] and $\{x \in \mathbb{R}^n : ||x||_2 = 1\} = \partial B(0,1)$ is compact in $\mathbb{R}^n$ as closed and bonded.

Remark that $\forall x \in \mathbb{R}^n$, $||Ax||_2 \leq ||A||_{2,2}||x||_2$. Indeed, $\forall x \in \mathbb{R}^n \backslash \{0\}$, $||Ax||_2 = ||A\frac{x}{||x||}||_2 \; ||x||_2 \leq ||A||_{2,2} \; ||x||_2$. Define $a_k = A^k$. Then observe that $(Id_N - A)\sum_{k=0}^{n} A^k = \sum_{k=0}^{n} A^k - \sum_{k=1}^{n+1} A^k = Id_N - A^{n+1}$.

Assume that $||A||_{2,2} < 1$. Then, $Id_N - A$ is invertible, i.e. $ker(Id_N - A) = \{0\}$. Indeed, $\forall x \in ker(Id_N - A)$, i.e. $(Id_N - A)x = 0$, which gives $x = Ax$ and so $||x||_2 = ||Ax||_2 \leq ||A||_{2,2}\,||x||_2$, which contradicts that $||A||_{2,2} < 1$ unless $||x||_2 = 0$, i.e. $x = 0$. It follows that $\sum_{k=0}^{n} A^k = (Id_N - A)^{-1}(Id_N - A^{n+1})$. We will show that

$\sum_{k=0}^{\infty} A^k = (Id_N - A)^{-1} \quad \forall\, x \in X \quad .$

$$\begin{aligned}||(\sum_{k=0}^{n} A^k - (Id - A)^{-1})x||_2 &= ||(Id - A)^{-1}(Id_N - A^{n+1} - Id_N)x||_2 \\ &= ||(Id - A)^{-1}A^{n+1}\,x||_2 \\ &\leq ||(Id - A)^{-1}||_{2,2}\; ||A^{n+1}x||_2 \\ &\leq ||(Id - A)^{-1}||_{2,2}\; ||A||_{2,2}\; ||A^n x||_2 \\ &\leq \dots \\ &\leq ||(Id - A)^{-1}||_{2,2}\, ||A||_{2,2}^{n+1}\; ||x||_2\end{aligned}$$

, which gives $|| \sum_{k=0}^{n} A^k - (Id - A)^{-1}|||_{2,2} \leq ||(Id - A)^{-1}||_{2,2}\, ||A||_{2,2}^{n+1} \to 0$ as $n \to \infty$, since $||A||_{2,2} < 1$. Therefore, $\sum_{k=0}^{\infty} A^k = (Id - A)^{-1}$.

Proposition 2.1

1. $\forall\ k_0' > k_0,\quad \sum_{k=k_0'}^{\infty} a_k$ converges if and only if $\sum_{k=k_0}^{\infty} a_k$ converges (therefore, we will often without loss of generality take $k_0 = 0$ or 1 in what follows)

2. If $\sum_{k=k_0}^{\infty} a_k$ converges, then $\lim_{k\to\infty}\ a_k = 0$.

Proof.

1. $\forall\, n \geq k_0',\quad \sum_{k=k_0}^{n} a_k = a_{k_0} + \cdots + a_{k_0'-1} + \sum_{k=k_0'}^{n} a_k$; hence, $\sum_{k=k_0}^{\infty} a_k$ converges if and only if $\sum_{k=k_0'}^{\infty} a_k$ converges.

2. $\forall n \geq k_0,\quad a_n = s_n - s_{n-1} = \sum_{k=k_0}^{n} a_k - \sum_{k=k_0}^{n-1} a_k$; hence if $\lim_{n\to\infty}\ s_n = s$, then $\lim_{n\to\infty}\ a_n = s - s = 0$.

□

Proposition 2.2

If $\sum_{k=k_0}^{\infty} a_k$ and $\sum_{k=k_0}^{\infty} b_k$ converges, then $\forall\, \lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$), $\sum_{k=k_0}^{\infty} (\lambda a_k + \mu b_k)$ converges and $\sum_{k=k_0}^{\infty} (\lambda a_k + \mu b_k) = \lambda \sum_{k=k_0}^{\infty} a_k + \mu \sum_{k=k_0}^{\infty} b_k$.

Proof. Follows from $\sum_{k=k_0}^{n} (\lambda a_k + \mu b_k) = \lambda \sum a_k + \mu \sum b_k \quad \forall\, n \geq k_0$. □

3 Cauchy Criterion for Convergence

Theorem 3.1: Cauchy Criterion for Convergence

If $(X, ||.||)$ is Banach, then $\sum_{k=k_0}^{\infty} a_k$ converges if and only if $\forall\, \varepsilon > 0, \quad \exists n_\varepsilon \geq k_0$ such that $\forall\, n' > n_\varepsilon$

$$|| \sum_{k=n+1}^{n'} a_k || < \varepsilon$$

Proof. If $(X, ||,||)$ is Banach, then $\sum_{k=k_0}^{\infty} a_k$ converges if and only if $(s_n)_{n \geq k_0}$, where $s_n = \sum_{k=k_0}^{n} a_k$ is Cauchy in $(X, ||.||)$, i.e. $\forall\, \varepsilon > 0, \quad \exists\, n_\varepsilon \geq k_0$ such that $\forall\, n' > n \geq n_\varepsilon, \quad ||s_{n'} - s_n|| = \sum_{k=n+1}^{n'} a_k < \varepsilon$ □

Definition 3.1: W

e say that the series $\sum_{k=k_0}^{\infty} a_k$ **converges absolutely** if $\sum_{k=k_0}^{\infty} ||a_k||$ converges in $(\mathbb{R}, |.|)$.

Remark 3.1

$\tilde{s}_n = \sum_{k=k_0}^{n} ||a_k||$ is increasing, hence $(s_n)_{n \geq k_0}$ converges if and only if it is bounded above. Moreover, if $(\tilde{s}_n)_{n \geq k_0}$ is bounded above, then $\sum_{k=k_0}^{\infty} ||a_k|| = \sup_{n \geq k_0} \tilde{s}_n$.

Proposition 3.1

If $(X, ||,||)$ is Banach, then for every series $\sum_{k=k_0}^{\infty} \ a_k$ in (X, d), if $\sum_{k=k_0}^{\infty} a_k$ converges absolutely, then $\sum_{k=k_0}^{\infty} a_k$ converges.

Moreover, $\forall\, n \geq k_0, \quad || \sum_{k=n}^{\infty} a_k || \leq \sum_{k=n}^{\infty} ||a_k||$.[a]

[a] We will see in Assignment 5 that the converse is true: if for every $\sum a_k$, $\sum a_k$ converges absolutely, then $\sum a_k$ converges, then the space is Banach

Proof. Since $\sum_{k=k_0}^{\infty} a_k$ converges absolutely, $\forall\, \varepsilon > 0, \quad \exists\, n_\varepsilon \geq k_0$ such that $\forall\, n \geq n_\varepsilon$

$$\left| \underbrace{\sum_{k=k_0}^{\infty} ||a_k|| - \sum_{k=k_0}^{n} ||a_k||}_{\sum_{k=n+1}^{n'} ||a_k||} \right| < \varepsilon$$

It follows that

$$\forall\, n' > n \geq n_\varepsilon \left|\left| \sum_{k=n+1}^{n'} a_k \right|\right| \leq \sum_{k=n+1}^{n'} ||a_k|| \leq \sum_{k=n+1}^{\infty} ||a_k|| < \varepsilon$$

. Hence, by using the Cauchy Criterion, since $(X, ||.||)$ is Banach, we obtain that $\sum_{k=k_0}^{\infty} a_k$ converges.

Moreover,

$$\forall\, n' > n \geq k_0, \quad \Big|\Big|\sum_{k=n}^{n'} a_k\Big|\Big| \leq \sum_{k=n}^{n'} ||a_k|| \leq \sum_{k=n}^{\infty} ||a_k||$$

Since $\sum_{k=n}^{\infty} a_k$ converges, by continuity of $||.||$, it follows that

$$\Big|\Big|\sum_{k=n}^{\infty} a_k\Big|\Big| \leq \sum_{k=n}^{\infty} ||a_k|| \qquad \left(\Big|\Big|\sum_{k=n}^{\infty} a_k\Big|\Big| = \Big|\Big|\lim_{n'\to\infty}\left(\sum_{k=n}^{n'} a_k\right)\Big|\Big| = \lim_{n'\to\infty}\Big|\Big|\sum_{k=n}^{n'} a_k\Big|\Big|\right)$$

□

Proposition 3.2

If $(X, ||.||)$ is Banach and $\sum_{k=1}^{\infty} a_k$ converges absolutely, then for every bijective function $f : \mathbb{N} \to \mathbb{N}$, the series $\sum_{k=1}^{\infty} a_{f(k)}$ converges to the same sum as $\sum_{k=1}^{\infty} a_k$.
We call $\sum_{k=1}^{\infty} a_{f(k)}$ a rearrangement of $\sum_{k=1}^{\infty} a_k$

Example 3.1

$$\sum_{k=0}^{\infty} 2^{-k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \dots$$

Proof. Since $\sum_{k=1}^{\infty} a_k$ converges absolutely, $\forall\, \varepsilon > 0, \quad \exists\, n_\varepsilon \in \mathbb{N}$ such that $\forall\, n \geq n_\varepsilon$

$$\Bigg|\underbrace{\sum_{k=1}^{\infty} ||a_k|| - \sum_{k=1}^{\infty} ||a_k||}_{=\sum_{k=n+1}^{\infty} ||a_k||}\Bigg| < \varepsilon$$

Moreover, since $(X, ||.||)$ is Banach and $\sum_{k=1}^{\infty}$ converges absolutely, we have that $\sum_{k=1}^{\infty} a_k$ converges.

Define $N_\varepsilon = \max_{1\leq k\leq n_{\varepsilon/2}}], f^{-1}(k)$ so that $\{1, \dots, n_{\varepsilon/2}\} \subseteq f\Big(\{1, \dots, N_\varepsilon\}\Big)$, i.e. $\{f^{-1}(1), \dots, f^{-1}(n_{\varepsilon/2})\} \subseteq \{1, \dots, N_\varepsilon\}$.Then,

$$\Big|\Big|\sum_{k=1}^{N_\varepsilon} a_{f(k)} - \sum_{k=1}^{n_{\varepsilon/2} a_k}\Big|\Big| = \Big|\Big|\sum_{k\in K_\varepsilon} a_k\Big|\Big| \leq \sum_{k\in K_\varepsilon} ||a_k|| \leq \sum_{k=n_{\varepsilon/2}+1}^{\infty} ||a_k|| < \frac{\varepsilon}{2}$$

,

$$\text{where } K_\varepsilon = f\Big(\{1, \dots, N_\varepsilon\}\Big) \Big\backslash \{1, \dots, n_{\varepsilon/2}\} \subseteq \{n \in \mathbb{N} : n > n_{\varepsilon/2}\}$$

It follows that

$$\Big|\Big|\sum_{k=1}^{N_\varepsilon} a_{f(k)} - \sum_{k=1}^{\infty} a_k\Big|\Big| \leq \Big|\Big|\sum_{k=1}^{N_\varepsilon} a_{f(k)} - \sum_{k=1}^{n_{\varepsilon/2}}\Big|\Big| + \Big|\Big|\sum_{k=n_{\varepsilon/2}+1}^{\infty} a_k\Big|\Big| < \varepsilon$$

Therefore, $\sum_{kk=1}^{\infty} a_{f(k)}$ converges to $\sum_{k=1}^{\infty} a_k$. $\square$

4 Convergence for Series with Non-Negative Terms

4.1 Comparison Test

Proposition 4.1

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series in $(\mathbb{R}, |.|)$ such that $0 \leq a_k \leq b_k \quad \forall\, k \in \mathbb{N}$. Then,

1. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges
2. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges

Proof.

1. $\sum_{k=1}^{\infty} b_k$ converges, then $\sum b_k$ is bounded above $\Rightarrow \sum a_k$ is bounded from above since $a_k \leq b_k \quad \forall\, k \in \mathbb{N}$. So, $\sum a_k$ converges as well.

2. follows from 1)

$\square$

4.2 Root Test

Proposition 4.2

Let $\sum_{k=1}^{\infty} a_k$ be a series in $(\mathbb{R}, |.|)$ such that $a_k \geq 0 \quad \forall\, k \in \mathbb{N}$

1. If $\limsup_{k\to\infty} \sqrt[k]{a_k} < 1$, then $\sum_{k=1}^{\infty} a_k$ converges
2. If $\limsup_{k\to\infty} \sqrt[k]{a_k} > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges

Remark 4.1

If $\limsup_{k\to\infty} \sqrt[k]{a_k} = 1$, then this test does not give any information.

Proof. Let $r = \limsup_{k\to\infty} \sqrt[k]{a_k} = \lim_{n\to\infty} \sup_{k\geq n} \sqrt[k]{a_k}$

1. Assume that $r < 1$, then $\forall\, \varepsilon > 0, \quad \exists\, n_\varepsilon \in \mathbb{N}$ such that $\forall\, n \geq n_\varepsilon, \quad \sup_{k\geq n} \sqrt[k]{a_k} < r + \varepsilon$, which gives $\forall\, k \geq n_\varepsilon, \quad \sqrt[k]{a_k} < r_\varepsilon$, i.e. $a_k < (r+\varepsilon)^k$.

Writing $\varepsilon \in (0, 1-r)$ (with $1-r > 0$ since $r < 1$), we obtain that $\sum_{k=1}^{\infty}(r+\varepsilon)^k$ converges. Hence, by the Comparison Test, it follows that $\sum_{k=1}^{\infty} a_k$ converges.

2. Assume that $r > 1$. Then, $\exists\ n_1 \in \mathbb{N}$ such that $\forall\ n \geq n_1$, $\sup_{k\geq n} \sqrt[k]{a_k} > 1$; hence $\forall\ n \geq n_1$, $\exists\ k \geq n$ such that $\sqrt[k]{a_k} > 1$, i.e. $a_k > 1$. In other words, $\exists\ (a_{n_k})_{k\in\mathbb{N}}$ subsequence such that $a_{n_k} > 1 \quad \forall\ k \in \mathbb{N}$. It follows that $\sum_{n=n}^{n_k} a_n \geq \sum_{j=1}^{k} a_{n_j} \geq \sum_{j=1}^{k} 1 = k$ (since $a_n \geq 0 \quad \forall\ n$). As $k \to \infty$, we have $k \to \infty$, so the series diverges.

$\square$

4.3 Ratio test

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Proposition 4.3

Let $\sum_{k=0}^{\infty} a_k$ be a series in $(\mathbb{R}, |.|)$ such that $\exists\ k_0 \in \mathbb{N}, \quad \forall\ k \geq k_0, \quad a_k > 0$. Then,

1. If $\limsup_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$, then $\sum_{k=0}^{\infty} a_k$ converges

2. If $\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} > 1$, then $\sum_{k=0}^{\infty} a_k$ diverges

Remark 4.2

- If $\limsup_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$ and $\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$, then the test gives no information.[a]

- $\liminf$ cannot be replaced by $\limsup$ in 2).

[a] i.e. $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$

Example 4.1

Let $a_k = \begin{cases} 2^{-k} & \text{if } k \text{ is odd} \\ 3^{-k} & \text{if } k \text{ is even} \end{cases}$. Then, $\sqrt[k]{a_k} = \begin{cases} \frac{1}{2} & \text{if } k \text{ is odd} \\ \frac{1}{3} & \text{if } k \text{ is even} \end{cases}$, ; hence, $\sum_{k=0}^{\infty} a_k$ converges by the root test. However, $\frac{a_{k+1}}{a_k} = \begin{cases} \frac{2}{3} & \text{if } k \text{ is odd} \\ \frac{3}{2} & \text{if } k \text{ is even} \end{cases}$, and so $\limsup_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{3}{2} > 1$.

Proof:

1. Let $r = \limsup_{k\to\infty} \frac{a_{k+1}}{a_k}$. Then, $\forall\, \varepsilon > 0, \quad \exists\, k_2 \in \mathbb{N}$ such that $\forall\, k \geq k_2$, $\sup_{k\geq\infty} \frac{a_{k+1}}{a_k} < r + \varepsilon$. Since $r < 1$, we can choose $\varepsilon > 0$ so that $r + \varepsilon < 1$. Since $\sup k \geq k_2 \frac{a_{k+1}}{a_k} < r + 1$, we obtain that $a_{k+1} \leq (r+\varepsilon)a_k \quad \forall\, k > k_2$ which implies $a_k \leq (r+\varepsilon)a_{k-1} \leq (r+\varepsilon)^2 a_{k_2} \leq \cdots \leq (r+\varepsilon)^{k-k_2} a_{k_\varepsilon} \quad \forall\, k \geq k_2$. Since $r + \varepsilon < 1$, we obtain that $\sum_{k=k_\varepsilon}^{\infty}(r+\varepsilon)^k$ converges. Hence, by comparison, $\sum_{k=0}^{\infty} a_k$ converges.

2. Since $\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} > 1$, $\exists\, k_1 \in \mathbb{N}$, $\forall\, n \geq k_1$, $\inf_{k\geq n} \frac{a_{k+1}}{a_k} > 1$, which implies that $\forall\, k \geq k_1$, $\frac{a_{k+1}}{a_k} > 1$, i.e. $a_{k+1} > a_k$, which implies $\forall\, k \geq k_1$, $a_k \geq a_{k-1} \geq \cdots \geq a_{k_1}$. Therefore, $\sum_{k=0}^{\infty} a_k$ diverges since $a_k \not\to 0$ as $k \to \infty$. □

5 Convergence of Series with Negative Terms

This is also the **Case of series which converge but do not converge absolutely**

5.1 Alternating Series Test

Proposition 5.1

Let $\sum_{k=0}^{\infty}(-1)^k a_k$ be a series in $(\mathbb{R}, |.|)$ such that $\lim_{k\to\infty} \ a_k = 0$ and $\forall\, k \in \mathbb{N}, \quad 0 \leq a_{k+1} \leq a_k$. Then, $\sum_{k=0}^{\infty}(-1)^k a_k$ converges and $0 \leq \sum_{k=0}^{\infty}(-1)^k a_k \leq a_0$.

Example 5.1

$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ converges since $\lim_{k\to\infty} \ \frac{1}{k} = 0$ and $0 \leq \frac{1}{k+1} < \frac{1}{k} \quad \forall\, k \in \mathbb{N}$

Proof. Observe that $\forall\, k \in \mathbb{N}$

$$\begin{aligned} s_{2n} &= \sum_{k=0}^{2n}(-1)^k a_k = a_0 - a_1 + a_2 - \ldots - a_{2n-1} + a_{2n} \\ &= a_0 - (a_1 - a_2) - (a_3 - a_4) - \ldots - (a_{2n-1} - a_{2n}) \\ &= a_0 - \sum_{k=1}^{n}(a_{2k-1} - a_{2k}) \quad \text{ and } a_{2k-1} - a_{2k} \geq 0 \text{ by assumption} \end{aligned}$$

Furthermore, $s_{2n+1} = (a_0 - a_1) + (a_2 - a_3) + \cdots + (a_{2n} - a_{2n+1}) = \sum_{k=0}^{n}(a_{2k} - a_{2k+1})$ with $(a_{2k} - a_{2k+1}) \geq 0$ by assumption.

Moreover, $s_{2n+1} = s_{2n} - a_{2n+1} \le s_{2n}$.

It follows that $\begin{cases} (s_{2n})_n \text{ is decreasing and bounded from below by } 0 \\ (s_{2n+1})_n \text{ is increasing and bounded from above by } a_0 \\ 0 \le s_{2n+1} \le s_{2n} \le a_0 \end{cases}$

Hence, $\exists\, s, s' \in [0, a_0]$ such that $\lim_{n\to\infty} s_{2n} = s$ and $\lim_{n\to\infty} s_{2n+1} = s'$. Since $s_{2n+1} = s_{2n} - a_{2n+1}$ and $\lim_{n\to\infty} a_{2n+1} = 0$ by assumption, it follows that $s = s'$, so the series converges to $s = s' \in [0, a_0]$. $\square$

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Proposition 5.2

Let $\sum_{k=0}^{a_k}$ and $\sum_{k=0}^{\infty} b_k$ be series in $(\mathbb{R}, |.|)$ such that $\sum_{k=0}^{\infty} a_k$ converges **absolutely** and $\sum_{k=0}^{\infty} b_k$ converges.
For each $n \ge 0$, define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

Then, $\sum_{n=0}^{\infty} c_n$ converges, and $\sum_{n=0}^{\infty} c_n = \Big(\sum_{k=0}^{\infty} a_k\Big)\Big(\sum_{k=0}^{\infty} b_k\Big)$.

The series $\sum_{n=0}^{\infty} c_n$ is called the Cauchy product of $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$.

Example 5.2

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges absolutely $\forall\, x \in \mathbb{R}$. Indeed, $\frac{\frac{|x|^{k+1}}{(k+1)!}}{\frac{|x|^k}{k!}} = \frac{|x|}{k+1} \to 0$ as $k \to \infty$; hence the ratio test applies.

$$e^x e^y = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}$$

, since $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ and $\sum_{k=0}^{\infty} \frac{y^k}{k!}$ converge absolutely.

Remark 5.1

We can now also define $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, $ch\ x = \frac{e^x+e^{-x}}{2}$, $sh\, x = \frac{e^x-e^{-x}}{2}$ and derive trigonometric formulas from the formula $e^{x+y} = e^x e^y$.

Counter-example: in case $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ converge but do not converge absolutely:

$\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}$ converges by the alternating series test, but does not converge absolutely (Assignment

5, Q4)

$$\sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \cdot \frac{(-1)^{n-k}}{\sqrt{n-k+1}} = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}$$

$$(k+1)(n-k+1) = -k^2 + nk + n + 1 \qquad \text{which attains its maximum at } k = \frac{n}{2}$$

$$\Rightarrow \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}} \geq \sum_{k=0}^{n} \frac{1}{\sqrt{(\frac{n}{2}+1)(n+\frac{n}{2}+1)}} \ldots \qquad \text{other steps but too long}$$

Therefore, $\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \frac{(-1)^{n-k}}{\sqrt{n-k+1}}$.

Proof of the Cauchy Product Proposition:

We obtained that $\forall\, N \in \mathbb{N}$, $\sum_{n=0}^{N} c_n = \sum_{n=0}^{N} \sum_{k=0}^{n} a_k b_{n-k}$.

Since $\sum_{k=0}^{\infty} b_k$ converges, we can write $\sum_{k=0}^{N-n} b_k = \sum_{k=0}^{\infty} b_k \; - \; \sum_{k=N-n+1}^{\infty} b_k$, which gives

$$\sum_{n=0}^{N} c_n = \Big(\sum_{n=0}^{N} a_n\Big)\Big(\sum_{k=0}^{\infty} b_k\Big) \; - \; \sum_{n=0}^{N} a_n \sum_{k=N-n+1}^{\infty} b_k$$

It remains that $\lim_{N\to\infty} \left(\sum_{n=0}^{N} a_n \sum_{k=N-n+1}^{\infty} b_k\right) = 0$.

Since $\sum_{k=0}^{\infty} b_k$ converges, $\forall\, \varepsilon > 0$, $\exists\; n_\varepsilon \in \mathbb{N}$ such that $\forall\; n \geq n_\varepsilon$, $\left|\sum_{k=0}^{\infty} b_k - \sum_{k=0}^{n} b_k\right| < \varepsilon$ (since $\sum_{k=N-n+1}^{\infty} b_k = \sum_{k=0}^{\infty} - \sum_{k=0}^{n} b_k$).Then, we obtain

$$\begin{aligned}
\forall\, N \geq n_\varepsilon,\quad \left|\sum_{n=0}^{N} a_n \sum_{k=N-n+1}^{\infty} b_k\right| &\leq \sum_{n=0}^{N} |a_n| \cdot \left|\sum_{k=N-n+1}^{\infty} b_k\right| \\
&= \underbrace{\sum_{n=0}^{N-n_\varepsilon} |a_n|}_{\infty} \cdot \underbrace{\left|\sum_{k=N-n+1}^{\infty} b_k\right|}_{<\varepsilon} + \sum_{n=N-n_\varepsilon+1}^{N} |a_n| \cdot \left|\sum_{k=N-n+1}^{\infty} b_k\right| \\
&\leq \sum_{n=0}^{\infty} |a_n| \qquad \text{since } n \leq N - n_\varepsilon \Rightarrow N - n + 1 \geq n_\varepsilon + 1
\end{aligned}$$

We now write: $$\sum_{n=N-n_\varepsilon+1}^{N} |a_n| \cdot \left|\sum_{k=N-n+1}^{\infty} b_k\right| = \sum_{n=1}^{n_\varepsilon} |a_N - n_\varepsilon + n| \cdot \left|\sum_{k=n_\varepsilon-n+1}^{\infty} b_k\right| \qquad \text{which } \to 0 \text{ as } k \to \infty$$

It follows that

$$\limsup_{N\to\infty} \left|\sum_{n=0}^{N} a_n \sum_{k=N-n+1}^{\infty} b_k\right| \leq \Big(\sum_{n=0}^{\infty} |a_n|\Big)\varepsilon$$

The, letting $\varepsilon \to 0$, we obtain that $\lim_{N\to\infty} \left(\sum_{n=0}^{N} a_n \sum_{k=N-n+1}^{\infty} b_k\right) = 0$. $\square$

6 Power Series

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Definition 6.1: Power Series

Let $(a_k)_{k\geq 0}$ be a sequence in $(\mathbb{R}, |.|)$ (or $(\mathbb{C}, |.|)$) and $x_0 \in \mathbb{R}$.

Then, the function $x \mapsto \sum_{k=0}^{\infty} a_k(x-x_0)^k$ (at points where it converges) is called a power series

Proposition 6.1

Let $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series in $(\mathbb{R}, |.|)$.

Define $R = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{a_k}}$, with the convention that $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$. Then, $\forall\, x \in \mathbb{R}$:

1. If $|x-x_0| < R$, then $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ converges absolutely

2. If $|x-x_0| > R$, then $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ diverges.

Remark 6.1

- If $R = 0$, then (2) holds $\forall\, x \neq x_0$

- If $R = \infty$, then (1) holds $\forall\, x$.

Example 6.1

$\sum_{k=0}^{\infty} x^k \qquad R = 1, \qquad\qquad e^x = \sum_{k=1}^{\infty} \frac{x^k}{k!}, \qquad R = \infty$

Proof. 1. If $|x-x_0| < R$, then $\limsup_{k\to\infty} \sqrt[k]{|a_k(x-x_0)^k|} < 1$, hence the root test gives that $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ converges absolutely.

2. If $|x-x_0| > R$, then $\limsup_{k\to\infty} \sqrt[k]{|a_k(x-x_0)^k|} > 1$.
Hence $\exists\ k_1 \in \mathbb{N},\ \forall\ n \geq k_1,\ \sup_{k\geq n} \sqrt[k]{|a_k(x-x_0)^k|} > 1$, i.e. $\left(a_k(x-x_0)^k\right)_k$ does not converge to 0 and $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ diverges.

$\square$

Proposition 6.2

Let $f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series in $(\mathbb{R}, |.|)$ and $R > 0$ be its radius of convergence. Then, f is **infinitely differentiable** in $(x_0 - R,\ x_0 + R)$ and for every $N \in \mathbb{N}$, the N^{th} derivation $f^{(N)}$ of f is given by

$$\begin{aligned} f^{(N)}(x) &= \sum_{k=N}^{\infty} k(k-1)\dots(k-N+1)a_k(x-x_0)^{k-N} \\ &= \sum_{k=0}^{\infty}(k+N)(k+N-1)\dots(k+1)a_{k+N}(x-x_0)^k \end{aligned}$$

In particular, $f^{(N)}(x_0) = N!\, a_n$, i.e. $a_N = \frac{1}{N!}f^{(N)}(x_0)$.

Moreover, the radius of convergence of this new series is R as well.[a]

[a] See MATH 249/466 for the complex analogue.

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Proof. It suffices to prove the result for $N = 1$. The case $N \geq 2$ follows by induction. Without loss of generality, by composing with $x \mapsto x + x_0$, we may assume that $x_0 = 0$.

Consider $\sqrt[k]{k|a_k|} \geq \sqrt[k]{|a_k|}$; hence, the radius of convergence of $\sum_{k=1}^{\infty} k\ a_k(x-x_0)^k$ is $\leq R$ (we will show later that it is also $\geq R$).

$$\begin{aligned} \forall\, x \in (-R, R), \quad t \neq 0, \quad \forall\, n \geq 2, \quad & \frac{1}{t}\Big(\sum_{k=0}^{n} a_k(x+t)^k - \sum_{k=0}^{n} a_k x^k\Big) - \sum_{k=1}^{n} k a_k x^{k-1} \\ &= \sum_{k=2}^{n} a_k \frac{(x+t)^k - x^k - kx^{k-1}t}{t} \\ &= \sum_{k=2}^{n} \frac{k(x+s)^{k+1}t - kx^{k-1}t}{t} \quad \text{for some } s \in (0,t) \text{ by Mean V} \\ &= \sum_{k=2}^{n} k a_k\Big((x+s)^{k-1} - x^{k-1}\Big) \\ &= \sum_{k=2}^{n} k a_k(k-1)(s+s')^{k-2}s \quad \text{for some } s' \in (0,s) \text{ by MVT} \end{aligned}$$

$$\begin{aligned} \Rightarrow \left|\frac{1}{t}\Big(\sum_{k=0}^{n} a_k(x+t)^k - \sum_{k=0}^{n} a_k x^k\Big) - \sum_{k=1}^{n} k a_k x^{k-1}\right| &= \left|\sum_{k=1}^{n} k(k-1)a_k(x+s')^{k-2}s\right| \qquad (*_1) \\ &\leq \sum_{k=1}^{n} k(k-1)|a_k|\big(|x|+|t|\big)^{k-2} + 1 \end{aligned}$$

Since $x \in (-R, R)$, letting $\varepsilon > 0$ be small enough so that $|x| < (1-\varepsilon)R$, and t be small enough so that $|x|+|t|<(1-\varepsilon)R$, which gives

$$\limsup_{k\to\infty}\frac{\sqrt[k-2]{|a_k|}(|x|+|t|)}{\sqrt[k-2]{|a_k|}(|x|+|t|)^{k-2}}<1-\varepsilon$$

, thus giving $\exists\ k_\varepsilon\geq 2$ such that $\forall\ k\geq k_\varepsilon$, $\begin{cases} k(k-1)|a_k|\big(|x|+|t|\big)^{k-2}\leq k(k-1)(1-\varepsilon)^{k-2} \\ |ka_kx^{k-1}|\leq k|a_k|\ |x|^{k-1}\leq k(1-\varepsilon)^{k-2}|x|\leq k(1-\varepsilon)^{k-2}R \end{cases}$ $(*_2)$

$\sum k(k-1)(1-\varepsilon)^{k-2}$ and $\sum k(1-\varepsilon)^{k-2}$ converge by ratio test:

$\dfrac{(k+1)k(1-\varepsilon)^{k-1}}{k(k-1)(1-\varepsilon)^{k-2}}=\dfrac{k+1}{k-1}(1-\varepsilon)\to_{k\to\infty}11-\varepsilon<1$ and $\dfrac{(k+1)(1-\varepsilon)^{k-1}}{k(1-\varepsilon)^{k-2}}=\dfrac{k+1}{k}(1-\varepsilon)\to_{k\to\infty}1-\varepsilon<1$

Hence, $\displaystyle\sum_{k=2}^{\infty}k(k-1)|a_k|\big(|x|+|t|\big)^{k-2}$ converges and $\displaystyle\sum_{k=1}^{\infty}ka_kx^{k-1}$ converges absolutely

[2] Letting $n\to\infty$ in $(*_1)$ and $(*_2)$, we obtain

$$\left|\frac{1}{t}\Big((f(x+t)-f(x)\Big)-\sum_{k=1}^{\infty}ka_kx^{k-1}\right|\leq|t|\sum_{k=2}^{\infty}k(k-1)(1-\varepsilon)^{k-2}$$

Letting $t\to 0$, it follows that

$$\lim_{t\to 0}\left(\frac{1}{t}\Big(f(x+t)-f(x)\Big)-\sum_{k=1}^{\infty}ka_kx^{k-1}\right)=0$$

i.e. $f'(x)=\sum_{k=1}^{\infty}ka_kx^{k-1}$. □

Proposition 6.3

1. $\forall\ x\in\mathbb{R},\quad \frac{d}{d(x)}[e^x]=e^x$

2. $x\mapsto e^s$ is increasing bijective from $\mathbb{R}$ to $(0,\infty)$, $\lim_{x\to\infty}e^x=\infty$ and $\lim_{x\to-\infty}e^x=0$

Proof. 1. Since the radius of convergence of $e^x=\sum_{k=0}^{\infty}\frac{s^k}{k!}$ is ∞, the previous result gives that $x\mapsto e^x$ is infinitely differentiable and

$$\frac{d}{dx}[e^x]=\sum_{k=1}^{\infty}\frac{x^{k-1}}{(k-1)!}=\sum_{k=0}^{\infty}\frac{x^k}{k!}=e^x$$

[2] In particular, this implies that the radius of convergence is $\geq R$

2. $\forall\, x \geq 0, \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} > 0$ since $x^k > 0 \quad \forall\, k$
   $\forall\, x < 0, \quad e^x = \frac{e^0}{e^{-x}} = \frac{1}{e^{-x}} > 0$ since $e^{-x} > 0$

   hence, $\frac{d}{dx}[e^x] = e^x > 0$, which implies that $x \mapsto e^x$ is strictly increasing and, in particular, also injective.

   $\forall\, x \geq 0, \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \geq 1 + x$ and $\lim_{x\to\infty} (1+x) = \infty$; hence, $\lim_{x\to\infty} e^x = \infty$.

   $\lim_{x\to-\infty} e^x = \lim_{x\to-\infty} \frac{1}{e^{-x}} = \lim_{x\to\infty} \frac{1}{e^x} = 0$ since $\lim_{x\to\infty} e^x = \infty$.

By using the Intermediate Value Theorem, together with the continuity of $x \mapsto e^x$, $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$, we obtain that $x \mapsto e^x$ is surjective from $\mathbb{R}$ to $(0,\infty)$ (Exercise of Analysis I).

$\square$

Remark 6.2

We can now define $\ln : (0,\infty) \to \mathbb{R}, \quad x \mapsto y$ such that $e^y = x$ and $\forall\, p > 0, \quad (0,\infty) \to (0,\infty), \quad x \mapsto x^p = e^{p\ln x}$.

$$\text{Moreover,} \qquad \ln(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k \quad \forall\, x \in (0,2)$$

Indeed, by observing that $\frac{d}{dx}\left[\frac{(-1)^{k-1}}{k}(x-1)^k\right] = (-1)^{k-1}(x-1)^{k-1}$, we obtain that $\frac{d}{dx}\left[\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k\right] = \sum_{k=1}^{\infty}(-1)^{k-1}(x-1)^{k-1} = \sum_{k=1}^{\infty}(1-x)^{k-1} = \frac{1}{1-(1-x)}$ and the radius of convergence of $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k$ is the same as that of $\sum_{k=1}^{\infty}(1-x)^{k-1}$, namely 1.

Now, let y be such that $e^y = x$, and observe that

$$\frac{d}{dy}\left[\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(e^y-1)^k - y\right] =$$

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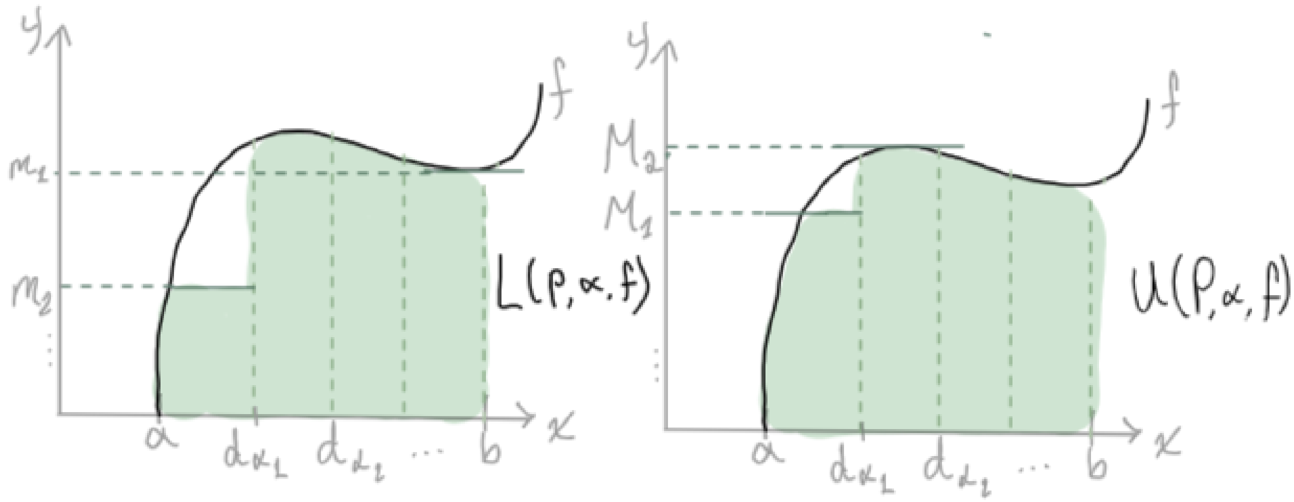
Let $\alpha : [a,b] \to \mathbb{R}$ be monotone and $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$. For every partition $P, \ \{x_0, \ldots, x_n\}$ of $[a,b]$, we defined $L(P,\alpha,f) = \sum_{i=1}^{n} m_i \Delta\alpha_i$ and $\cup(P,\alpha,f) = \sum_{i=1}^{n} M_i \Delta\alpha_i$, where $m_i = \inf_{x_{i-1}\leq x\leq x_i} f(x), \quad M_i = \sup_{x_{i-1}\leq x\leq x_i} f(x)$ and $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$.

We then define $\int_{-a}^{b} f d_\alpha = \sup L(P,f,\alpha)$ and $\int_{a}^{-b} f d_\alpha = \inf \cup(P,f,\alpha)$ (we also denote $\int_{-a}^{b} f(x) d_\alpha(x)$



and $\int_{a}^{-b} f(x)d_\alpha(x)$). $\int_{-a}^{b} fd_\alpha$ and $\int_{a}^{-b} fd_\alpha$ are called the **lower** and **upper Riemann-Stieltjes integrals** of f over $[a,b]$ with respect to α.

In the case where $\int_{-a}^{b} fd_\alpha = \int_{a}^{-b} fd_\alpha$, we denote $\int_{a}^{b} fd_\alpha = \int_{-a}^{b} fd_\alpha = \int_{a}^{-b} fd_\alpha$ (also denoted $\int_{a}^{b} f(x)d_\alpha(x)$) and we say that f is **Riemann-Stieltjes integrable** over $[a,b]$ with respect to α and $\int_{a}^{b} fd_\alpha$ is called the *R-S* integral of f over $[a,b]$ *w.r.t.* α. The set of all bounded functions $f : [a,b] \to \mathbb{R}$ which are *R-S* integrable over $[a,b]$ *w.r.t.* α is denoted $R\Big([a,b],\alpha\Big)$ or $R(\alpha)$.

If $\alpha(x) = x$, we say Riemann integrable/integral instead of Riemann-Stieltjes integrable/integral and we drop the α in the notations $R\Big([a,b]\Big)$, $U(P,f)$, $L(P,f)$, $\int_{-a}^{b} f(x)dx$, etc.

In the case where $\alpha(x) \neq x$, the value of f in $[x_{i-1},x_i]$ are weighted by $\Delta\alpha_i$ instead of $\Delta x_i = x_i - x_{i-1}$: the more α increased in $[x_{i-1},x_i]$,, the larger the weight.

Theorem 6.1: Geometric Interpretation in the case where $\alpha(x) = x$ and $f(x) \geq 0$

$\int_{-a}^{b} fdx =$ supremum of the areas of the regions between the axis $y = 0$ and the graphs of piecewise constant function smaller than f on $[a,b]$.

$\int_{a}^{-b} fdx =$ infimum of the areas of the regions between the axis $y = 0$ and the graphs of the piecewise constant function larger than f on $[a,b]$.

Remark 6.3

1. α is monotone; hence, it is bounded: $\forall\, x \in [a,b], \quad \alpha(a) \leq \alpha(x) \leq \alpha(b)$.

2. $$\Big(\inf_{a\leq x\leq b} f(x)\Big) \overbrace{\sum_{i=1}^{n} \Delta\alpha_i}^{=\alpha(b)-\alpha(a)} \leq L\big(P,f,\alpha\big) \leq U\big(P,f,\alpha\big) \leq \Big(\sup_{a\leq x\leq b} f(x)\Big) \overbrace{\sum_{i=1}^{n} \Delta\alpha_i}^{=\alpha(b)-\alpha(a)}$$
since $\inf_{a\leq x\leq b} f(x) \leq m_i \leq M_i \leq \sup_{a\leq x\leq b} f(x)$ and $\Delta\alpha_i \geq 0$, hence
$$\int_{-a}^{b} f dx, \quad \int_{a}^{-b} f dx \;\in\; \Big[a\Big(\inf_{a\leq x\leq b} f(x)\Big)\big(\alpha(b)-\alpha(a)\big),\; \Big(\sup_{a\leq x\leq b} f(x)\Big)\big(\alpha(b)-\alpha(a)\big)\Big]$$
and if $f \in R\Big([a,b],\alpha\Big)$, then $\Big(\inf_{a\leq x\leq b} f(x)\Big)\big(\alpha(b) - \alpha(a)\big) \leq \int_a^b f dx \leq \Big(\sup_{a\leq x\leq b} f(x)\Big)\big(\alpha(b)-\alpha(a)\big)$

Definition 6.2: Refinements

1. Let P and P^* be two partitions of $[a,b]\Big|f$, $P \subseteq P^*$, then we say that P^* is a **refinement** of P.

2. Let P_1 and P_2 be two partitions of $[a,b]$. Then, we say that $P_1 \bigcup P_2$ is the **common refinement** of P_1 and P_2.

Proposition 6.4

If P^* is a refinement of P, then $L\big(P^*,f,\alpha\big) \geq L\big(P,f,\alpha\big)$ and $U\big(P^*,f,\alpha\big) \leq U\big(P,f,\alpha\big)$.

Proof. We prove the case where P^* consists in exactly one more element than P. The general case is then obtained by induction.
Let $x^* \in (a,b)$ be such that $P^* = P \cup \{x^*\}$. Then, $\exists\, i \in \{1,\ldots,n\}$ such that $x_{i-1} < x^* < x_i$. Then,
$$L\big(P^*,f,\alpha\big) - L\big(P,f,\alpha\big) = m_i^*\Delta\alpha_i^* + m_{i+1}^*\Delta\alpha_{i+1}^* - m_i\Delta\alpha_i$$
where $m_i = \inf_{x_{i-1}\leq x\leq x^*} f(x)$, $m_i^* = \inf_{x_{i-1}\leq x\leq x^*} f(x)$, $m_{i+1}^* = \inf_{x*\leq x\leq x_i} f(x)$, $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$, $\Delta\alpha_i^* = \alpha(x^*) - \alpha(x_{i-1})$ and $\Delta\alpha_{i+1}^* = \alpha(x_i) - \alpha(x^*)$.

Observe that $m_i^* \geq m_i$ and $m_{i+1}^* \geq m_i$; since $[x_{i-1},x^*] \subseteq [x_{i-1},x_i]$ and $[x^*,x_i] \subset [x_{i-1},x_i]$. Moreover, $\Delta\alpha_i^* \geq 0$ and $\Delta\alpha_{i+1}^* \geq 0$ (because α is monotone. Hence, $L\big(P^*,f,\alpha\big) - L(P,f,\alpha) \geq$

$m_i\big(\Delta a_i^* + \Delta\alpha_{i+1}^* - \Delta\alpha_i\big) = m_i\big(\Delta\alpha_i - \Delta\alpha_i\big) = 0.$

Similarly, we obtain that $U(P^*, f, \alpha) \leq U(P, f, \alpha)$. ☐

Proposition 6.5

$$\int_{-a}^{b} f d\alpha \leq \int_{a}^{-b} f d\alpha$$

Proof. $\forall\, P_1, P_2$ partitions of $[a, b]$, letting $P^* = P_1 \cup P_2$,

$$\begin{aligned} L(P, f, \alpha) &\leq L(P^*, f, \alpha) && \text{(By previous result)} \\ &\leq U(P^*, f, \alpha) && \text{(since inf} \leq \text{sup)} \\ &\leq U(P_2, f, \alpha) && \text{(By previous result)} \end{aligned}$$

It follows by taking the supremum on P_1 and the infimum on P_2 that $\int_{-a}^{b} f d\alpha \leq \int_{a}^{-b} f d\alpha$ ☐

Proposition 6.6

$f \in R\Big([a, b], \alpha\Big)$ if and only if $\forall\ \varepsilon > 0$, there exists a partition P_ε of $[a, b]$ such that $U(P_2, f, \alpha) - L(P_2, f, \alpha) \leq \varepsilon$.

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Proof. Assume that $f \in R\big([a, b], \alpha\big)$. Then, $\forall\, \varepsilon > 0, \quad \exists\, P_\varepsilon, P_\varepsilon'$ partitions of $[a, b]$ such that

$$\underbrace{\int_a^b f dx}_{=\int_a^b f d\alpha \text{ since } f \in R\big([a,b],\alpha\big)} \quad -\frac{\varepsilon}{2} < L(P_\varepsilon, f, \alpha) \leq \int_a^b f d\alpha = \int_a^b f dx \leq U(P_\varepsilon', f, \alpha) \leq \underbrace{\int_a^b f dx}_{=\int_a^b f d\alpha} + \frac{\varepsilon}{2}$$

Letting $P_\varepsilon^* = P_\varepsilon \cup P_\varepsilon'$, and noting that $\bigcup(P_\varepsilon^*, f, \alpha) \leq \bigcup(P_\varepsilon', f, \alpha)$ and $L(P_\varepsilon^*, f, \alpha) \geq L(P_\varepsilon, f, \alpha)$, we obtain

$$\begin{aligned} \bigcup(P_\varepsilon^*, f, \alpha) - L(P_\varepsilon^*, f, \alpha) &\leq \bigcup(P_\varepsilon', f, \alpha) - L(P_\varepsilon, f, \alpha) \\ &\leq \left(\int_a^b f d\alpha + \frac{\varepsilon}{2}\right) - \left(\int_a^b f d\alpha - \frac{\varepsilon}{2}\right) \\ &\leq \frac{\varepsilon}{2} - \left(\frac{-\varepsilon}{2}\right) = \varepsilon \end{aligned}$$

Conversely, assume that $\forall\, \varepsilon > 0, \quad \exists\, P_\varepsilon$ partitions of $[a, b]$ such that $U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon$. Since $L(P_\varepsilon, f\alpha) \leq \int_{-a}^{b} f d\alpha \leq \int_{a}^{-b} f d\alpha \leq U(P_\varepsilon, f, \alpha)$, we obtain $0 \leq \int_a^{-b} - \int_a^a f d_x \leq U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) \leq \varepsilon$. Letting $\varepsilon \to 0$, we obtain $\int_{-a}^{b} f d_\alpha = \int_a^{-b} f d\alpha$. ☐

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Proposition 6.7

Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be monotone on I. Then the set D of all points at which f is not continuous is countable.

Remark 6.4

This result together with the result we proved last week show that if α is continuous, then every monotone function is $R - S$ integrable over $[a, b]$ with respect to α.

Proof. For each $x \in D$, $\lim_{y\to x^-} f(y)$ and $\lim_{y\to x^+} f(y)$ exist (since f is monotone) but $\lim_{y\to x^-} f(y) <_{\neq} \lim_{y\to x^+} f(y)$ (since $x \in D$). By density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists\, q_x \in \left(\lim_{y\to x^-} f(y), \lim_{y\to x^+} f(y)\right)$. Moreover, $\forall\, x, y \in I,\quad x < y \Rightarrow q_x \neq q_y$ (with $<$ if f is increasing and $>$ if f is decreasing).

it follows that $D \to \mathbb{Q},\quad x \mapsto q_x$ is injective, hence D is countable $\square$

Example 6.2

Example of a function which is not R-S integrable where α is continuous:

$f(x) = \begin{cases} 0 & \text{if } x \in [a, b]\backslash\mathbb{Q} \\ 1 & \text{if } x \in [a, b] \cap \mathbb{Q} \end{cases}$. In this case, $\forall\ P$ partition of $[a, b]$, $m_1 = 0$ and $M_i = 1$ where $\Delta x_i > 0$, hence $L(P, f, \alpha) = 0$ and $U(P, f, \alpha) = \alpha(b) - \alpha(a)$ and so $\int_{-a}^{b} f d\alpha = 0$ and $\int_{a}^{-b} f d_x = \alpha(b) - \alpha(a) \neq 0$ if $\alpha(b) > \alpha(a)$.

Remark 6.5

This function, however, is ”Lebesgue” integral (see MATH 454).

Proposition 6.8

Let $a, b, c, d \in \mathbb{R}$ be such that $a < b$ and $c < d$, $f : [a, b] \to [c, d]$, $\phi : [c, d] \to \mathbb{R}$ and $\alpha : [a, b] \to \mathbb{R}$ be such that α is increasing on $[a, b]$. If ϕ is continuous on $[c, d]$ and $f \in R\big([a, b], \alpha\big)$, then $\phi \circ f \in R\big([a, b], \alpha\big)$.

Proof. Let $\varepsilon > 0$. By uniform continuity of ϕ (continuous compact set $[c, d]$) $\exists\ \delta_\varepsilon > 0$ such that $\forall\, x, y \in [c, d],\quad |x - y| < \delta_\varepsilon \quad\Rightarrow\quad |\phi(x) - \phi(y) < \lambda\varepsilon$, where λ is a constant to be chosen later. Since $f \in R\big([a, b], \alpha\big)$, $\exists\, P_\varepsilon$ partition of $[a, b]$ such that

$$U\big(P_\varepsilon, f, \alpha\big) - L\Big(P_\varepsilon, f, \alpha\Big) = \sum_{i=1}^{n_\varepsilon} (M_i - m_i)\Delta\alpha_i \le \delta_\varepsilon^2$$

We want to show that

$$U(P_\varepsilon, \phi\circ f, \alpha) - L(P_\varepsilon, \phi\circ f, \alpha) < \varepsilon$$

$$= \sum_{i=1}^{n_\varepsilon} \underbrace{\Big(\sup_{x_{i-1}\le x\le x_i} \phi\circ f(x) - \inf_{x_{i-1}\le x\le x_i} \phi\circ f(x)\Big)}_{=\omega_i} \Delta\alpha_i$$

- If $\omega_i \le \delta_\varepsilon$, then $\omega_i \le \lambda\varepsilon by$ $(*)$
- If $\omega_i > \delta_\varepsilon$, then $\omega_i \le M - m$ where $M = \max_{c\le x\le d}\phi(x)$ and $m = \min_{c\le x\le d}\phi(x)$

Then,

$$U(P_\varepsilon, \phi\circ f, \alpha) - L(P_\varepsilon, \phi\circ f, \alpha) = \lambda\varepsilon \underbrace{\sum_{i=1}^{n_\varepsilon}\alpha_i}_{\omega_i\le\delta_\varepsilon} + (M-m) \underbrace{\sum_{i=1}^{n_\varepsilon}\Delta\alpha_i}_{\omega_i>\delta_\varepsilon,\quad i.e.\ \omega_i/\delta_\varepsilon>1}$$

$$\le \lambda\varepsilon \underbrace{\sum_{i=1}^{n_\varepsilon}\Delta\alpha_i}_{=\alpha(b)-\alpha(a)} + (M-m) \underbrace{\sum_{i=1}^{n_\varepsilon}\frac{\omega_i}{\delta_\varepsilon}\Delta\alpha_i}_{\le 1/\delta_\varepsilon \sum_{i=1}^{n_\varepsilon}\omega_i\Delta\alpha_i \le \delta_\varepsilon^2/\delta_\varepsilon = \delta_\varepsilon}$$

By $(*_2)$

Now, choose $\lambda - \frac{1}{2\big(\alpha(b)-\alpha(a)\big)}$ (if $\alpha(b) > \alpha(a)$) and δ_ε small enough so that $\delta_\varepsilon \le \frac{\varepsilon}{2(M-m)}$ (if $M > m$). Then, $U(P_\varepsilon, \phi\circ f, \alpha) - L(P_\varepsilon, \phi\circ f, \alpha) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ □

Proposition 6.9

Let $a, b \in \mathbb{R}$ be such that $a < b$ and $\alpha : [a,b] \to \mathbb{R}$ be increasing on $[a,b]$. Then,

1. $\forall\, f, g \in R\big([a,b],\alpha\big),\quad f+g \in R\big([a,b],\alpha\big)$ and $\displaystyle\int_a^b (f+g)d\alpha = \int_a^b f d\alpha + \int_a^b$

Lecture 24 - 04/08/2025

7 Taylor's Theorem with Integral Remainder

Theorem 7.1: Taylor's Integral Remainder

Let $x \ne x_0 \in \mathbb{R}$ $I = \begin{cases} [x_0, x] & \text{if } x > x_0 \\ [x, x_0] & \text{if } x < x_0 \end{cases}$ and $f : I \to \mathbb{R}$ be $n+1$ times differentiable in $\mathring{I}$, n times continuously differentiable on I and $t \mapsto f^{(n+1)}(t)(x-t)^n$ is Riemann integrable over

I. Then

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)} x_0 (x-x_0)^k + \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt \quad \text{where} \quad \int_{x_0}^{x} = -\int_{x}^{x_0} \text{ if } x < x_0$$

Proof. In the first version of Taylor's Theorem, we defined $h(t) = g(t)(x-x_0)^{n+1} - g(x_0)(x-t)^{n+1}$, where

$$g(t) = f(x) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(t)(x-t)^k$$

and we obtained that $h(x) = h(x_0) = 0$ and $h'(t) = (n+1)g(x_0)(x-t)^n - \frac{1}{n!} f^{(n+1)}(t)(x-t)^n(x-x_0)^{n+1}$. Consider that h is continuous on I (since f is n times continuous differentiable on I)and that it is differentiable in $\mathring{I}$ (since f is $n+1$ times differentiable in I). Furthermore, h' is R-integrable over I (since $t \mapsto f^{(n+1)}(t)(x-t)^n$ is R-integrable).

Hence, the Fundamental Theorem of Calculus gives $0 = 0 - 0 = h(x) - h(x_0) = \int_{x_0}^{x} h' = g(x_0)(x-x_0)^{n+1} - \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt (x-x_0)^{n+1}$, which then gives $0 = (x-x_0)^{n+1}\Big(g(x_0) - \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt\Big)$. Thus, we get $g(x_0) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt = f(x) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k$ □

Definition 7.1: L

et $a \in \mathbb{R}$, $\;b \in (a,\infty)$ or $b = \infty$ and $\alpha : [a,b) \to \mathbb{R}$ be increasing on $[a,b]$. We say that $f : [a,b] \to \mathbb{R}$ is Riemann-Stieltjes integrable over $[a,b)$ with respect to α if $\forall\, s \in (a,b)$, $\;f \in R\Big([a,s],\alpha\Big)$ and $\lim_{s\to b}\left(\int_a^s f dx\right)$ $\exists$ and is finite. We denote $\int_{[a,b)} f dx = \lim_{s\to b}\left(\int_a^s f dx\right)$ and we call $\int_{[a,b)} f dc$ the $R-S$ integral of f over $[a,b)$ with respect to α. When $b = \infty$, we denote $\int_{[a,\infty)} f dx = \int_a^\infty f dx$. Similarly, we define $\int_{[a,b)} f dx$ and $\int_{-\infty}^{b} f da$

Proposition 7.1

Assume that $a > -\infty$, $b < \infty$ and $f \in R\big([a,b],\alpha\big)$.

1. If α is continuous at b, then $\int_{[a,b)} f dx = \int_a^b f dx$.
2. If α is continuous at a, then $\int_{(a,b]} f dx = \int_a^b f dx$.

Proof.

$$(1) \qquad \forall\, s \in (a,b), \quad \left|\int_a^b f dx - \int_a^s f dx\right| = \left|\int_s^b f dx\right| \le \int_s^b |f| dx \le \sup_{a \le x \le b} |f| \big(\alpha(b) - \alpha(a)\big)$$

, by continuity of α and since $f \in R\big((a,b],\alpha\big)$.

The proof for (2) is similar. $\square$

Example 7.1

$$(1) \qquad \int_1^\infty \frac{dx}{x^\alpha} \quad \alpha \in \mathbb{R}$$

8 Integral Convergence Test

Theorem 8.1: Integral Convergence Test

Let $n_0 \in \mathbb{N}$ and $f : [n_0, \infty) \to [0, \infty)$ be decreasing and such that $\lim_{x\to\infty} f(x) = 0$. Define

$$s_n = \sum_{k=n_0}^{n} f(k), \quad t_n = \int_{n_0}^{n} f \quad \text{and} \quad d_n = t_n - s_n$$

Then,

1. $0 \le f(n+1) \le d_{n+1} \le d_n \le f(n_0) \quad \forall\, n \ge n_0$
2. $d_n \to d \in [0, f(n_0))$
3. $\sum_{k=n_0}^{\infty} f(k)$ converges if and only if f is R integrable over $[n_0, \infty)$
4. $0 \le d_n - d \le f(n) \quad \forall\, n \ge n_0$

1. *Proof.*

$$\begin{aligned} d_{n+1} &= \sum_{k=n_0}^{n=1} f(k) - \int_{n_0}^{n} f \\ &= f(n+1) + \sum_{k=n_0}^{n} \overbrace{\Big(f(k) - \int_k^{k+1} f\Big)}^{k+1} && (\ge f(n+1)) \\ &= \int_k^{k+1} \big(\underbrace{f(k) - f(s)}_{>0}\big) ds && (\text{Since } f \text{ is decreasing}) \end{aligned}$$

and so we have

$$
\begin{aligned}
d_n &= \sum_{k=n_0}^{n} f(k) - \int_{n_0}^{n} f \\
&= f(n_0) + \sum_{k=n_0+1}^{n} \big(f(k) - \int_{k-1}^{k} f\big) \leq f(n_0) \\
&= \int_{k-1}^{k} \big(f(k) - f(s)\big) ds \leq 0
\end{aligned}
$$

$$
d_{n+1} - d_n = f(n+1) - \int_{n}^{n+1} f = \int_{n}^{n+1} \big(f(n+1) - f(s)\big) ds \leq 0
$$

$\square$

2. *Proof.* Since $(d_n)_{n\in\mathbb{N}}$ is monotone and bounded by 0 and $f(n_0)$, by (1), we obtain that $\lim_{n\to\infty} (d_n) = d$ for some $d \in [0, f(n_0)]$. $\square$

3. *Proof.* Since $f \geq 0$, we have that $x \mapsto \int_{n_0}^{x} f$ is increasing. Therefore, $\lim_{x\to\infty} \int_{n_0}^{x} f$ $\exists$ and is finite (i.e. f is $R-$integrable over $[n_0, \infty)$) iff $\lim_{n\to\infty} t_n$ $\exists$ and is infinite. Since, by (2), $\lim_{n\to\infty} (s_n - t_n) = d \in \mathbb{R}$, we obtain that $(s_n)_{n\in\mathbb{N}}$ (i.e. $\sum_{k=n_0}^{n} f(k)$) converges iff $(t_n)_n$ converges.
   Hence, $\sum_{k=n_0}^{\infty} f(k)$ converges iff f is R-integrable over $[n_0, \infty)$. $\square$

4. *Proof.*

$$
\begin{aligned}
\forall\, n' > n \geq n_0, \quad d_n - d_{n'} &= \sum_{k=n}^{n'-1} (d_k - d_{k+1}) \\
&= \sum_{l=n}^{n'-1} \int_{k}^{k+1} \big(\overbrace{f(s) - f(k+1)}^{\in[0,\, f(k)-f(k+1)]}\big) ds \\
\text{Hence, } 0 \leq d_n - d_n' &\leq \sum_{k=n}^{n'-1} \big(f(k) - f(k+1)\big) \qquad\qquad \text{(Since } f \text{ is decreasing)}
\end{aligned}
$$

Since $\lim_{n\to\infty} d_n = d$ and $\lim_{n\to\infty} \underbrace{f(n)}_{=0} = f(n) - f(n')$, it follow s that $0 \leq d_n - d \leq f(n)$. $\square$

—— THE MATERIAL COVERED BY THE FINAL EXAM STOPS HERE

A Change of Variable Formula

Theorem A.1: Variable Change

Let $a, b, A, B \in \mathbb{R}$ be such that $a < b$ and $A < B$, $\quad \alpha : [a, b] \to \mathbb{R}$ be increasing, $f : [a, b] \to \mathbb{R}$ be bounded and $\varphi : [A, B] \to [a, b]$ be strictly increasing and surjective. Then, $f \in R\big([a, b], \alpha\big)$ iff $f \circ \varphi \in R\big([A, B], \varphi \circ \alpha\big)$. Moreover, if $f \in R\big([a, b], \alpha\big)$, then

$$\int_a^b f dx = \int_A^B f \circ \varphi d(\alpha \circ \varphi)$$