MATH 251 – Honours Algebra 2
Final Exam Material

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Chapter 1

Vector Spaces

Definition 1.0.1: Vector Space

A vector space over $\mathbb{F}$ is a set V together with 2 operations

$$(u, v) \mapsto u + v$$

$$(\alpha, v) \mapsto \alpha v$$

Satisfying

1. V with addition is an abelian group
2. $1_{\mathbb{F}} v = v$
3. $(\alpha\beta)v = \alpha(\beta v)$
4. $(\alpha + \beta)v = \alpha v + \beta v$
5. $\alpha(u + v) = \alpha u + \alpha v$

Definition 1.0.2: Subspace

A subspace W of V is a subset satisfying

1. $0_V \in W$
2. $w_1 + w_2 \in W$
3. $\alpha w \in W$

Note that W is a vector space with the same operations as V.

Definition 1.0.3: Algebraic Sum

The algebraic sum of U, W over $\mathbb{F}$ is

$$U + W = \{\alpha u + \beta w : u \in U,\, w \in W,\, \alpha, \beta \in \mathbb{F}\}$$

Definition 1.0.4: External Direct Sum

The external direct sum of U, W over $\mathbb{F}$ is

$$U \boxplus W = \{(u, w) : u \in U,\, w \in W\}$$

Definition 1.0.5: Span$(v_i)_{i\in I}$

$\text{Span}(v_i)_{i\in I} = \{\sum_{i\in I} \alpha_i v_i : \alpha_i \in \mathbb{F}\}$

Definition 1.0.6: Spanning Family

S is a spanning family if $\text{Span}(S) = V$.

Definition 1.0.7: Minimal Spanning Family

S is a minimal spanning family if it is spanning for index set I but, for every $I' \subsetneq I$, $\text{Span}(v_i)_{i\in I'} \neq V$.

Definition 1.0.8: Linearly Dependent and Independent Family

S is linearly dependent if $\exists\, \alpha_1, ..., \alpha_n \in \mathbb{F}$ not all 0 and $\{i_1, ..., i_n\} \subset I$ satisfying

$$\sum_{j=1}^{n} \alpha_j v_{i_j} = 0$$

If not, S is linearly independent.

Definition 1.0.9: Maximal Linearly Independent Family

If S is linearly independent and for any $v_* \in V$, the family $(v_i)_{i\in I\cup\{*\}}$ is linearly dependent.

Theorem 1.0.10

The Span of a family is a subspace

Proof. □

Chapter 2

Bases

Definition 2.0.1: Basis

S is a basis of V if it satisfies one of the following conditions

1. Minimal Spanning
2. Maximal Linearly Independent
3. $v \in V$ can be represented uniquely as a linear combination of S

Definition 2.0.2: Dimension

The dimension of a vector space is the cardinality of its basis

Theorem 2.0.3: Equivalent Definitions of a Basis

The following are equivalent

1. Minimal Spanning
2. Maximal Linearly Independent
3. $v \in V$ can be represented uniquely as a linear combination of S
4. Linearly Independent and Spanning

Proof. □

Lemma 2.0.4: Steinitz Substitution Lemma

Let $A = (v_1, ..., v_n)$ be linearly independent and let $B = (w_1, ..., w_m)$ be linearly independent with $m \geq n$. Then we can reorder the elements of B such that $(v_1, ..., v_n, w_{n+1}, ..., w_m)$ is linearly independent.

Proof. □

Corollary 2.0.5

The cardinality of a linearly independent set cannot exceed the cardinality of a basis

Proof. □

Corollary 2.0.6

The dimension of a vector space is well-defined

Proof. □

Corollary 2.0.7

Every linearly independent set can be completed to a basis and the dimension of a subspace is bounded by the dimension of the ambient space

Definition 2.0.8: $[v]_B$

Let $B = (b_i)$ be a basis of V. Then any $v \in V$ can be written uniquely as $v = \sum \alpha_i b_i$. We denote

$$[v]_B = (\alpha_1, ..., \alpha_n)$$

Theorem 2.0.9: Basis Changes

Let B, C be bases of V. Define

$$_C M_B = ([b_1]_C ... [b_n]_C)$$

then,

1. $[v]_C = {}_C M_B [v]_B$ only for $_C M_B$ as defined
2. $_B M_B = I$
3. $_D M_B = {}_D M_C \, {}_C M_B$
4. $_B M_C^{-1} = {}_C M_B$

Proof. □

Corollary 2.0.10

A matrix is invertible if and only if its columns form a basis

Proof. □

Chapter 3

Linear Maps

Definition 3.0.1: Linear Map

Let V, W be vector spaces over $\mathbb{F}$. A linear map $T : V \to W$ is a function satisfying

1. $T(v_1 + v_2) = T(v_1) + T(v_2)$
2. $T(\alpha v) = \alpha T(v)$

Definition 3.0.2: Image and Kernel

1. $\text{Im} T = \{w \in W : \exists v \in V \text{ s.t. } T(v) = w\}$
2. $\ker T = \{v \in V : T(v) = 0_W\}$

Corollary 3.0.3

The image and kernel of a linear map are subspaces of the target and source space respectively

Proof. □

Corollary 3.0.4

A linear map is injective if and only if its kernel is trivial

Proof. □

Definition 3.0.5: Linear Isomorphism

A linear map T is an isomorphism if it is bijective

Theorem 3.0.6: Isomorphism to $\mathbb{F}^n$

Every finite-dimensional vector space is isomorphic to $\mathbb{F}^n$

Proof. □

Theorem 3.0.7: Kernel and Image

Let $T : V \to W$ be a linear map where V is finite-dimensional. Then,

$$\dim V = \dim \ker T + \dim \mathrm{Im} T$$

Proof. □

Theorem 3.0.8: Matrix Representing a Linear Map

Let V, W, U be vector spaces over $\mathbb{F}$ of dimensions n, m, l with bases B, C, D and $T : V \to W$ and $R : W \to U$ be a linear map. Then,

1. There is a unique matrix ${}_C[T]_B \in \mathcal{M}_{m\times n}(\mathbb{F})$ (the matrix representing T) such that,
$$[Tv]_C = {}_C[T]_B[v]_B$$
2. If $R : V \to W$ is another linear map, Then
   (a) ${}_C[R + T]_B = {}_C[R]_B + {}_C[T]_B$
   (b) ${}_C[\alpha T]_B = \alpha\, {}_C[T]_B$
3. For every matrix $M \in \mathcal{M}_{m\times n}(\mathbb{F})$ there is a linear map $T : V \to W$ such that ${}_C[T]_B = M$
4. ${}_D[R \circ T]_B = {}_D[R]_C\, {}_C[T]_B$

Proof. □

Corollary 3.0.9

$L(V, W) \cong \mathcal{M}_{m\times n}(\mathbb{F})$

Proof. Follows directly from the theorem □

Definition 3.0.10: Inner Direct Sum

The inner direct sum of V, W both over $\mathbb{F}$ $V \oplus W$ is the algebraic sum of the two given that $V \cap W = \{0\}$

Corollary 3.0.11

V is the inner direct sum of $U, W \subset V$ if and only if $U \oplus W \cong U \boxplus W$

Proof. □

Definition 3.0.12: Projection

A projection is a linear map $T : V \to V$ such that $T \circ T = T$

Theorem 3.0.13: Inner Direct Sum Definition of a Projection

1. Let U, W be subspaces of V such that $V = U \oplus W$. Define the map $T : V \to V$ by $T(v) = u$. If $v = u + w$ where $u \in U$ and $w \in W$, then T is a projection with $\text{Im}T = U$ and $\ker T = W$
2. Let $T : V \to V$ be a projection and $\text{Im}T = U$ and $\ker T = W$. Then $V = U \oplus W$ and $T(u + w) = u$ for $u \in U$ and $w \in W$

Definition 3.0.14: Quotient Space

Let $U \subset V$ be a subspace. The quotient group V/U derived from equivalence relation $v \sim v'$ if $v = v' + u$ for some $u \in U$ is a vector space called the quotient space under the following operations

1. $[v] + [w] = [v + w]$
2. $\alpha[v] = [\alpha v]$

Chapter 4

Determinant

Definition 4.0.1: Permutation

A permutation of $\{1, 2, ..., n\}$ is a bijection

$$\sigma : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$$

Definition 4.0.2: Inversion

A pair $\{i, j\}$ with $i < j$ is an inversion of σ if $\sigma(i) > \sigma(j)$

Definition 4.0.3: Sign

$$\text{sgn}(\sigma) = (-1)^{|I(\sigma)|}$$

where $I(\sigma)$ is the set of all the inversions of σ

Definition 4.0.4: Transposition

σ is a transposition if there are $k < l \in \{1, ..., n\}$ such that $\sigma(i) = i \forall i \neq k, l$ and $\sigma(k) = l$, $\sigma(l) = k$

Definition 4.0.5: m-cycle

σ is an m-cycle if there are distinct $i_1, i_2, ..., i_m \in \{1, 2, ..., n\}$ such that

1. $\sigma(i) = i$ for $i \neq i_j$
2. $\sigma(i_j) = i_{j+1}$ from $1 \leq j < m$
3. $\sigma(i_m) = i_1$

Corollary 4.0.6

sgn$(\sigma \circ \pi)$ =sgn(σ)sgn(π)

Proof. ☐

Theorem 4.0.7: Existence and Uniqueness of Determinant

There is a unique function called the determinant

$$\det : \mathcal{M}_{n\times n}(\mathbb{F}) \to \mathbb{F}$$

satisfying

1. $\det(v_1 \cdots \alpha v_i \cdots v_n) = \alpha \det(v_1 \cdots v_i \cdots v_n)$
2. $\det(v_1 \cdots v_i + v_i' \cdots v_n) = \det(v_1 \cdots v_i \cdots v_n) \det(v_1 \cdots v_i' \cdots v_n)$
3. $\det(v_1 \cdots v_i \cdots v_j \cdots v_n) = 0$ if $v_i = v_j$
4. $\det(e_1 \cdots e_n) = 1$

Proof. ☐

Corollary 4.0.8

Functions satisfying axioms (1)-(3) satisfy

$$\det(v_1 \cdots v_i + \alpha v_j \cdots v_j \cdots v_n) = \det(v_1 \cdots v_i \cdots v_j \cdots v_n)$$

Theorem 4.0.9: Cauchy's Theorem

If $A, B \in \mathcal{M}_{n\times n}(\mathbb{F})$, then

$$\det(AB) = \det(A) \det(B)$$

Proof. ☐

Corollary 4.0.10

A matrix is invertible if and only if its determinant is nonzero

Definition 4.0.11: Minor

Let $A \in \mathcal{M}_{n\times n}(\mathbb{F})$. For $i, j = 1, \cdots, n$, the ij-minor A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} row and j^{th} column.

Definition 4.0.12: Cofactor

the ij-cofactor A^{ij} is given by

$$A^{ij} = (-1)^{i+j} \det(A_{ij})$$

Theorem 4.0.13: Laplace Theorem

For any i, we have

$$\det(A) = \sum_{j=1}^{n} a_{ij} A^{ij}$$

Equivalently, for any j, we have

$$\det(A) = \sum_{i=1}^{n} a_{ij} A^{ij}$$

Definition 4.0.14: Adjoint Matrix

$$\mathrm{Adj}(A) = (c_{ij}),\, c_{ij} = A^{ji}$$

Lemma 4.0.15: Cramer's Rule

Consider a non-homogeneous linear system of n equations and n unknowns

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

represented by $Ax = b$ with $A = (a_{ij})$. Assume that $\det(A) \neq 0$. Then there is a unique solution x to the system. Let A_i be the matrix obtained by replacing the i^{th} column of A by b. Then

$$x_i = \frac{\det(A_i)}{det(A)}$$

Definition 4.0.16: Orientation

?? Two ordered bases B and B' of V have the same orientation if the determinant of the change of basis matrix ${}_{B'}M_B$ has satisfies $\det({}_{B'}M_B)$

Definition 4.0.17: Isomorphism preserving orientation

??A linear isomorphism $T : V \to W$ that maps positively oriented bases of V to positively oriented bases of W

Definition 4.0.18: Wronskian

Let $f_1(t), \cdots, f_n(t) : \mathbb{R} \to \mathbb{R}$ $n-1$ times differentiable. The Wronskian is defined as

$$W(f_1, \cdots, f_n)(t) = \det \begin{pmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{pmatrix}$$

Definition 4.0.19: Row and Column Rank

Let $A = (a_{ij}) \in \mathcal{M}_{m\times n}$. The column rank of A is the dimension of the image of A or the dimension of the span of the columns of A.

$$\mathrm{rk}_C A = \dim \mathrm{Im}(A) = \dim \mathrm{Span}\{ \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} \}$$

The row rank of A is the dimension span of the rows of A.

$$\mathrm{rk}_R A = \dim \mathrm{Span}\{ \begin{pmatrix} a_{11} \cdots a_{1n} \end{pmatrix}, \cdots, \begin{pmatrix} a_{m1} \cdots a_{mn} \end{pmatrix} \}$$

Corollary 4.0.20

$\mathrm{rk}_C A$=$\mathrm{rk}_R A$

Proof. □

Chapter 5

Dual Spaces

Definition 5.0.1: Linear Functional

Let V be a vector space over $\mathbb{F}$, the a linear functional is a linear map

$$\phi : V \to \mathbb{F}$$

Definition 5.0.2: Dual Space

Let V be a vector space over $\mathbb{F}$, then the dual space of V: V^* is the set of linear functionals on defined on V.

$$V^* = \text{Hom}(V, \mathbb{F})$$

Theorem 5.0.3: Existence of a Dual Basis

Let $B = (b_1, \cdots, b_n)$ be a basis of V. Define $b_i^* \in V^*$ such that $b_i^*(b_j) = \delta_{ij}$, then $B^* = (b_1^*, \cdots, b_n^*)$ forms a basis of V^*.

Proof. □

Definition 5.0.4: Annihilator

Let $U \subset V$ be a subspace. The anihilator of V is

$$V^* \supset U^\perp = \{f \in V^* : f(u) = 0 \forall u \in U\}$$

Lemma 5.0.5: Annihilator Properties

1. $U^\perp$ is a subspace of V^*.
2. If $U \subset W \subset V$ are both subspaces, then $W^\perp \subset U^\perp$
3. $\dim U^\perp = \dim V - \dim U$

Proof. □

Corollary 5.0.6

The annhihilator of $U + W$ is the intersection of the annihilator of U and W

$$(U + W)^{\perp} = U^{\perp} \cap W^{\perp}$$

Proof. □

Definition 5.0.7: Dual Map

Given a linear map $T : V \to W$, the dual map $T^* : W^* \to V^*$ is the map satisfying for $f \in W^*$

$$T^*(f)(v) = f(T(v))$$

Corollary 5.0.8

Let B, C be bases of V, W, then

$${}_{B^*}[T^*]_{C^*} = {}_{C}[T]_{B}^{t}$$

Chapter 6

Inner Products

Definition 6.0.1: Inner Product

An inner product of a vector space V over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ is a function $\langle \cdot, \cdot \rangle = V \times V \to \mathbb{F}$ satisfying

1. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
2. $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$
3. $\langle v, w \rangle = \overline{\langle w, v \rangle}$
4. $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$

Definition 6.0.2: Norm

$$||v|| = \sqrt{\langle v, v \rangle}$$

Definition 6.0.3: Distance

$$d(v, w) = ||v - w||$$

Lemma 6.0.4: Pythagorean Lemma

If $\langle v, w \rangle = 0$, then

$$||v + w||^2 = ||v||^2 + ||w||^2$$

Proof. □

Theorem 6.0.5: Cauchy-Schwarz Inequality

For every $v, w \in V$, we have

$$|\langle v, w\rangle| \leq ||v||\,||w||$$

with equality if and only if v, w are linearly dependent

Proof. □

Lemma 6.0.6: Properties of the norm

1. $||v|| \geq 0$ with equality if and only if $v = 0$
2. $||\alpha v|| = |\alpha|\,||v||$
3. $||v + w|| \leq ||v|| + ||w||$

Proof. □

Definition 6.0.7: Hermitian Matrix

$A = (a_{ij}) \in \mathcal{M}_{n\times n}(\mathbb{F})$ is hermitian if $a_{ij} = \overline{a_{ji}}$.

$$A^* \equiv \overline{A^t} = A$$

Definition 6.0.8: Orthogonality

Let V be an inner product space, then $v, w \in V$ are orthogonal $v \perp w$ if $\langle v, w\rangle = 0$

Definition 6.0.9: Orthogonal Basis

A basis $(u_i)_{i=1}^n$ of an inner product space is orthogonal if $\forall j \neq k\ u_j \perp u_k$.

Definition 6.0.10: Orthonormal Basis

A basis $(u_i)_{i=1}^n$ is orthonormal if it is orthogonal and $||u_i|| = 1\,\forall i$

Theorem 6.0.11: Existence and Uniqueness of the Orthogonal Projection

Let $U \subset V$ be a finite-dimensional subspace with an orthonormal basis $(u_i)_{i=1}^n$. Then for each $v \in V$, There exists a unique $u \in U$ called the orthogonal projection of v onto U such that $v - u \perp U$ and

$$u = \sum_{i=1}^{n} \langle v, u_i\rangle\, u_i$$

Proof. □

Corollary 6.0.12

Let $(u_i)_{i=1}^n$ be an orthonormal basis of V, then each $v \in V$ is equal to its projection onto $U = V$.

$$v = \sum_{i=1}^{n} \langle v, u_i \rangle u_i$$

Corollary 6.0.13

$\langle v, w \rangle = \langle \sum_{i=1}^n \langle v, u_i \rangle u_i, \sum_{i=1}^n \langle w, u_i \rangle u_i \rangle = \sum_{i=1}^n \langle v, u_i \rangle \overline{\langle w, u_i \rangle}$

Corollary 6.0.14

$$||v||^2 = \langle v, v \rangle = \sum_{i=1}^{n} |\langle v, u_i \rangle|^2$$

Corollary 6.0.15

The isomorphism $T : V \to \mathbb{F}^n$ defined by $T(v) = [v]_B$ satisfies

$$\langle v, w \rangle = \langle T(v), T(w) \rangle_{\text{std}}$$

Theorem 6.0.16: Garm-Schmidt Process

Let $(b_1, \cdots, b_n)$ be a basis of an inner product space V . Then there exists an orthonormal basis $(u_1, \cdots, u_n)$ of V such that for each k we have $\text{Span}(u_1, \cdots, u_k) = \text{Span}(b_1, \cdots, b_k)$.

Proof. □

Definition 6.0.17: Orthogonal Subspace

Let $U \subset V$, then the orthogonal complement to U is

$$U^\perp = \{v \in V : \langle v, u \rangle = 0 \, \forall u \in U\}$$

Corollary 6.0.18

If U is a finite-dimensional subspace of inner product space V, then $V = U \oplus U^\perp$

Proof. □

Lemma 6.0.19: Minimum Distance

Let $U \subset V$ be a finite-dimensional subspace of an inner product space V, then the orthogonal projection of a vector $v \in V$ to U is the closest vector to v in U.

Chapter 7

Eigenvalues and Eigenvectors

Definition 7.0.1: Eigenvalue

Let $T : V \to V$ be a linear map. A scalar $\lambda \in \mathbb{F}$ is an eigenvalue of T if there is a non-zero vector $v \in V$ such that

$$T(v) = \lambda v$$

Definition 7.0.2: Eigenvector

A vector $v \in V$ is eigenvector of T if there is a scalar $\lambda \in \mathbb{F}$ such that

$$T(v) = \lambda v$$

Definition 7.0.3: Similar Matrices

$A, A' \in \mathcal{M}_{n\times n}$ are similar $A \sim A'$ if there is an invertible $M \in \mathcal{M}_{n\times n}$ such that

$$A' = M^{-1}AM$$

Corollary 7.0.4

Similar matrices have the same eigenvalues

Proof. □

Definition 7.0.5: Characteristic Polynomial

Let V be a finite-dimensional vector space over $\mathbb{F}$ and operator $T : V \to V$. The characteristic polynomial is defined as

$$\Delta_T(t) = \det(t\,\mathrm{Id} - T) = \det(tI - {}_B[T]_B)$$

Corollary 7.0.6

$\Delta_A(t) = t^n - \operatorname{tr} A\, t^{n-1} + \cdots + (-1)^n \det(A)$

Proof. □

Corollary 7.0.7

The following are equivalent:

1. λ is an eigenvalue of A
2. The matrix $\lambda I - A$ is not ivertible
3. $\Delta_A(\lambda) = 0$

Proof. □

Definition 7.0.8: Eigenspace

Let $T : V \to V$ be a linear map. Then the eigenspace of $\lambda \in \mathbb{F}$ is

$$E_\lambda = \ker(\lambda \operatorname{Id} - T) = \{v \in V : T(v) = \lambda v\}$$

Definition 7.0.9: Geometric Multiplicity

Let λ be an eigenvalue of T. Its geometric multiplicity is given by

$$m_g(\lambda) = \dim E_\lambda$$

Definition 7.0.10: Algebraic Multiplicity

Let λ be an eigenvalue of T. Its algebraic multiplicity $m_a(\lambda)$ is defined such as

$$\Delta_T(t) = (t - \lambda)^{m_a(\lambda)} g(t)$$

with $g(\lambda) \neq 0$

Corollary 7.0.11

If $T : V \to V$ with $\dim V = n$ has eigenvalue λ, then

$$1 \leq m_g(\lambda) \leq m_a(\lambda) \leq n$$

Proof. □

Definition 7.0.12: Diagonalisability

A linear map T is diagonalisable if there is a basis B such that

$$_B[T]_B = \text{diag}(\lambda_1, \cdots, \lambda_n)$$

for $\lambda_i \in \mathbb{F}$ not necessarily distinct.

Theorem 7.0.13: Equivalent Defintions of Diagonalisability

The following are equivalent:

1. T is diagonalisable
2. $\exists B$ a basis such that $_B[T]_B = \text{diag}(\lambda_1, \cdots, \lambda_n)$
3. The exists a basis of V consisting of eigenvectors of T
4. $m_g(\lambda) = m_a(\lambda)$ for every eigenvalue λ and $\Delta_T(t)$ factors into linear factors over $\mathbb{F}$

Proof. □

Corollary 7.0.14

The n^{th} Fibonacci number is given by

$$a_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$$

Proof. □

Chapter 8

Decomposition

Theorem 8.0.1: Caley-Hamilton Theorem

$$\Delta_A(A) = 0$$

Definition 8.0.2: Minimal Polynomial

The minimal polynomial of $A \in \mathcal{M}_{n\times n}(\mathbb{F})$ is the monic polynomial $m_A(t) \in \mathbb{F}[t]$ of minimal degree satisfying $m_A(A) = 0$.

Corollary 8.0.3

$m_A(t)$ divides $f(t)$ for any $f(t) \in \mathbb{F}[t]$ with $f(A) = 0$

Corollary 8.0.4

The polynomials Δ_A and m_A have the same irreducible factors. More precisely, for $A \in \mathcal{M}_{n\times n}$,

$$m_A(t)|\Delta_A(t)|m_a(t)^n$$

Definition 8.0.5: Invariant Subspaces

Let $T : V \to V$ be a linear map. we say that a subspace $W \subset V$ is T-invariant if $T(W) \subset W$. We can then consider the restriction

$$T_{|W} : W \to W$$

Theorem 8.0.6: Primary Decomposition Theorem

Let $T : V \to V$ be a linear map on a finite-dimensional vector space V with minimal polynomial $m_T(t) = f_1(t)^{n_1}...f_k(t)^{n_k}$ where f_i are irreducible factors. Let $W_i = \ker f_i^{n_i}(T)$, then

$$V = W_1 \oplus \cdots \oplus W_k$$

and $f_i^{n_i}$ is the minimal polynomial of $T_i = T_{|W_i}$

Corollary 8.0.7

$T : V \to V$ is diagonalisable if and only if the minimal polynomial of T factors into distinct linear terms over $\mathbb{F}$

$$m_T(t) = (t - \lambda_1) \cdots (t - \lambda_k)$$

where $\lambda_i \in \mathbb{F}$ are distinct.

Proof. □

Definition 8.0.8: Jordan Block

A Jordan block is the matrix $J_k \in \mathcal{M}_{k\times k}$ of the form

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

Definition 8.0.9: Jordan Form

A matrix has Jordan form if it is block diagonal with Jordan blocks

Theorem 8.0.10: Secondary Decomposition Theorem

$T : V \to V$ with $m_T(t) = (t - \lambda)^m$ is represented in an appropriate basis by a matrix in Jordan form with $\dim E_\lambda$ Jordan blocks $J_{n_i}(\lambda)$ of maximal n_i equal to m.

Definition 8.0.11: Nilpotent map

If $N : V \to V$ satisfies $N^m = 0$ for some m, then N is nilpotent.

Corollary 8.0.12

If $A = MJM^{-1}$ where $J = J_n(\lambda) = \lambda I + N$ with

$$J = J_n(\lambda) = \lambda I + \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} = \lambda I + N$$

Then

$$J^l = (\lambda I + N)^l = \sum_{i=0}^{l} \binom{l}{i} \lambda^{l-i} N^i$$

Note $N^2 = \begin{pmatrix} 0 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & 0 & 0 \end{pmatrix}$ and so on such that $N^n = 0$ (N is nilpotent). So we get that $J^l = \sum_{i=0}^{r} \binom{r}{i} \lambda^{r-i} N^i$ for $r = \min\{l, n\}$ and $A^l = MJ^lM^{-1}$.

Chapter 9

Adjoints

Definition 9.0.1: Hermitian Adjoint

Let $T : V \to V$ be a linear map. The hermitian adjoint is the linear map $T^* : V \to V$ satisfying

$$\langle Tv, w\rangle = \langle v, T^* w\rangle \, \forall v, w \in V$$

Theorem 9.0.2: Existence and Uniqueness of Hermitian Adjoint

A unique hermitian adjoint exists and, if B is an orthonormal basis, then

$$_B[T^*]_B = {}_B[T]_B^*$$

Proof. □

Definition 9.0.3: Self-Adjoint Map

T is self-adjoint if $T = T^*$, represented in an orthonormal basis B we get

$$_B[T]_B = {}_B[T]_B^*$$

Corollary 9.0.4

Self-adjoint maps have real eigenvalues and orthogonal eigenspaces.

Proof. □

Theorem 9.0.5: Diagonalisability of Self-Adjoint Maps

Let $T : V \to V$ be a self-adjoint map, the there exists a basis B such that $_B[T]_B$ is diagonal.

Proof. □

Definition 9.0.6: Unitary Map

A map M is unitary if

$$\langle Mv, Mw\rangle_{\text{std}} = \langle v, w\rangle_{\text{std}}\,\forall v, w \in \mathbb{C}^n$$

An equivalent definition is that M is unitary if and only if the matrix representation of M satisfies

$$M^*M = I$$

Definition 9.0.7: Symmetric Bilinear Form

A symmetric bilinear form is a function $(\cdot,\cdot) : V \times V \to \mathbb{R}$ satisfying

1. $(\cdot, v)$ is a linear map $\forall v \in V$
2. $(v, w) = (w, v)\,\forall v, w \in V$

Theorem 9.0.8: Principal Axis Theorem

Let A be a real symmetric matrix and $(\cdot,\cdot)$ the associated bilinear form

$$(v, w) = \begin{pmatrix} | \\ v \\ | \end{pmatrix} A \begin{pmatrix} - & w & - \end{pmatrix}$$

Then there is an orthonormal basis B such that for $[v]_B = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ we have

$$(v, w) = \sum_{i=1}^{n} \lambda_i v_i w_i$$

where λ_i's are eigenvalues of A with multiplicities as roots of Δ_A.

Proof. □

Definition 9.0.9: Positive Definiteness

A is positive definite if

$$v^* A v \geq 0$$

with equality if $v = 0$.

Theorem 9.0.10: Positive Definiteness of Hermitian Matrices

A hermitian matrix is positive definite if and only if all its eigen values are $\lambda_i > 0$.

Proof. □

Theorem 9.0.11: Sylvester's Theorem

If A is hermitian, then A is positive definite if and only if $\det(A_k) > 0$ for $k = 1, \cdots, n$ where A_k is the $k \times k$ top-left submatrix of A.

Proof. □

Lemma 9.0.12: Classification of Quadrics

Definition 9.0.13: Normal Map

A linear map $T : V \to V$ is normal if $T^*T = TT^*$.

Theorem 9.0.14: Diagonalisability of Normal Maps

Let $T : V \to V$ be a normal map on V a vector space over $\mathbb{C}$, then there is a basis B such that ${}_B[T]_B$ is diagonal.

Proof. □

Theorem 9.0.15: Spectral Theorem

Let $T : V \to V$ be a normal map on V a vector space over $\mathbb{C}$, then

$$T = \lambda_1 P_1 + \cdots + \lambda_k P_k$$

where λ_i's are eigenvalues of T and P_i's are orthogonal projections such that $P_1 + \cdots + P_k =$Id and $P_i P_j = 0$ for $i \neq j$

Proof. □