

# MATH 223: Linear Algebra

McGill University

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Condensed Notes

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$$\begin{array}{ccc} V_\beta & \xrightarrow{[T]_\beta} & V_\beta \\ \downarrow Q_\beta^\alpha & & \uparrow Q_\alpha^\beta \\ V_\alpha & \xrightarrow{[T]_\alpha} & V_\alpha \end{array}$$

The Similarity Transform (Theorem 68)

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# Foreword

This document contains my condensed notes for Jeremy Macdonald's MATH 223 class, taken in W2025. These notes were typeset manually by myself, in their entirety.

It is important to note that these are *condensed notes*, and do not account for a comprehensive treatment of MATH 223. These notes do not provide proofs of theorems, nor do they contain any discussion pertaining to the content of the course. They simply list out all covered theorems, propositions, and definitions. Thus, simply reading this document is unlikely to be sufficient practice to succeed in MATH 223.

For context, I had to take one of MATH 223 and MATH 247 for my Math minor. I chickened out, thinking MATH 223 was harder than it actually was, and took 223 instead of 247. Add-Drop has passed by the time I realized that I would be better served by 247, so I went to Prof. Tim Hoheisel's MATH 247 lectures, only attending 223 for exams. Since Hoheisel provided his own notes, I did not take any for 247. I wrote these up a few days before the 223 final, and proofread them in the summer. Babbling aside, the takeaway here is that you shouldn't be afraid to challenge yourself.

These notes consist of five discrete sections, following Prof. Macdonald's course outline. Additionally, each statement is one of the following, and is labeled as such.

**Definition:** An explanation of the mathematical meaning of a word or phrase.

**Theorem:** A significant statement that has been proven to be true.

**Proposition:** A less significant, but nonetheless important, proven-true statement.

**Lemma:** A simpler, true statement used in proving other true statements.

**Corollary:** A true statement that is a simple deduction from a theorem or proposition.

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# 1 Complex Numbers

## 1.1 Basics of $\mathbb{C}$

### Definition 1 (Field of Complex Numbers).

The field of complex numbers,  $\mathbb{C}$ , consists of all expressions of the form

$$a + bi,$$

where  $a, b \in \mathbb{R}$  and  $i$  is a symbol satisfying  $i^2 = -1$ .

### Definition 2 (Addition and Multiplication).

Let  $z = a_1 + b_1 i$  and  $\omega = a_2 + b_2 i$ . Define

$$z + \omega = (a_1 + a_2) + (b_1 + b_2) i, \quad z\omega = (a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i.$$

### Definition 3 (Conjugate and Modulus).

For  $z = a + bi$ ,

$$\bar{z} = a - bi \quad (\text{complex conjugate}), \quad |z| = \sqrt{a^2 + b^2} \quad (\text{absolute value or modulus}).$$

### Proposition 4.

If  $z = a + bi \neq 0$  (i.e.  $z \neq 0 + 0i$ ), then the number

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$$

is called the (multiplicative) inverse of  $z$  and satisfies

$$z z^{-1} = 1 = z^{-1} z.$$

### Definition 5 (Division).

For  $z, \omega \in \mathbb{C}$  with  $\omega \neq 0$ , define

$$\frac{z}{\omega} = z \omega^{-1}.$$

### Proposition 6.

Let  $z, w \in \mathbb{C}$ .

1.  $\overline{z + w} = \bar{z} + \bar{w}$ .

2.  $\overline{z w} = \bar{z} \bar{w}$ .

3.  $\overline{\bar{z}} = z$ .

4.  $z \bar{z} = |z|^2$ .

5.  $z \in \mathbb{R} \iff \bar{z} = z$ .

## 1.2 Polar Form and Equation

**Definition 7 (Complex Exponential Function).**

In the complex plane,  $e^{i\theta}$  is the unique number of modulus 1 and argument  $\theta$ . Equivalently,

$$e^{i\theta} = \cos \theta + i \sin \theta,$$

(Euler's formula).

**Theorem 8 (Fundamental Theorem of Algebra).**

Let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where  $a_i \in \mathbb{C}$ . Then  $p(z)$  factors into linear factors:

$$p(z) = a_n (z - r_1)(z - r_2) \cdots (z - r_n),$$

where each  $r_i \in \mathbb{C}$  (the  $r_i$  may repeat).

## 2 Vector Spaces

### 2.1 Vector Space Axioms

**Definition 9 (Field).**

A field is a set  $F$  equipped with two operations  $+$  and  $\cdot$  such that:

- $(F, +)$  is an abelian group.
- $(F \setminus \{0\}, \cdot)$  is an abelian group.
- Multiplication distributes over addition: for all  $a, b, c \in F$ ,

$$a(b + c) = ab + ac, \quad (a + b)c = ac + bc.$$

Examples include  $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ .  $\mathbb{Z}$  is not a field since, e.g.,  $\frac{2}{3} \notin \mathbb{Z}$ .

**Definition 10 (Vector Space).**

Let  $F$  be a field and  $V$  a set. A vector space over  $F$  is a pair  $(V, +)$  together with a scalar multiplication  $F \times V \rightarrow V$  satisfying the following eight axioms for all  $u, v, w \in V$  and  $a, b \in F$ :

1.  $u + v = v + u$ .
2.  $(u + v) + w = u + (v + w)$ .
3. There exists  $0 \in V$  with  $u + 0 = 0 + u = u$ .
4. For each  $u$  there is  $-u$  with  $u + (-u) = (-u) + u = 0$ .

5.  $a(u + v) = au + av$ .
6.  $(a + b)u = au + bu$ .
7.  $a(bu) = (ab)u$ .
8.  $1u = u$ , where  $1$  is the multiplicative identity in  $F$ .

**Proposition 11 (Basic Vector Space Properties).**

Let  $V$  be a vector space over a field  $F$ .

1. For all  $u, v, w \in V$ , if  $u + w = v + w$  then  $u = v$ .
2. The zero vector  $\vec{0}$  in  $V$  is unique.
3. For each  $u \in V$ , its additive inverse  $-u$  is unique.
4. For all  $u \in V$ ,  $0u = \vec{0}$ .
5. For all  $c \in F$ ,  $c\vec{0} = \vec{0}$ .
6. For all  $c \in F$  and  $u \in V$ ,  $(-c)u = c(-u) = -(cu)$ .

## 2.2 Linear Combinations and Subspaces

**Definition 12 (Linear Combination).**

Given vectors  $u_1, u_2, \dots, u_m \in V$  and scalars  $c_1, \dots, c_m \in F$ , any vector of the form

$$c_1 u_1 + c_2 u_2 + \cdots + c_m u_m$$

is called a linear combination of  $u_1, \dots, u_m$ .

**Definition 13 (Span).**

Let  $S = \{u_1, u_2, \dots, u_m\} \subseteq V$ . The span of  $S$  is

$$\text{span}(S) = \{c_1 u_1 + c_2 u_2 + \cdots + c_m u_m \mid c_i \in F\}.$$

If  $S = \emptyset$ , we define  $\text{span}(\emptyset) = \{0\}$ .

**Proposition 14.**

If  $A, B \in M_{m \times n}(F)$  and  $B$  is obtained from  $A$  by elementary row operations (EROs), then

$$\text{row}(A) = \text{row}(B).$$

**Proposition 15 (Facts About Spans).**

Let  $S \subseteq V$ . Then:

1. For all  $u, w \in \text{span}(S)$ ,  $u + w \in \text{span}(S)$  (closure under addition).

2. For all  $u \in \text{span}(S)$  and  $c \in F$ ,  $cu \in \text{span}(S)$  (closure under scalar multiplication).
3.  $\vec{0} \in \text{span}(S)$ .

**Definition 16 (Subspace).**

Let  $V$  be a vector space over a field  $F$ , and let  $W \subseteq V$ . We say  $W$  is a subspace of  $V$  (and write  $W \leq V$ ) if:

1. For all  $w_1, w_2 \in W$ ,  $w_1 + w_2 \in W$ .
2. For all  $w \in W$  and all scalars  $c \in F$ ,  $cw \in W$ .
3. The zero vector  $0 \in W$ .

**Theorem 17.**

Let  $A \in M_{m \times n}(F)$ ,  $b \in F^m$ , and let  $x \in F^n$  be the vector of variables. Let  $S$  be the set of all solutions to the linear system  $Ax = b$ . Then  $S$  is a subspace of  $F^n$  if and only if

$$b = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \vec{0} \quad (\text{i.e. the system is homogeneous}).$$

**Proposition 18.**

Subspaces are closed under forming linear combinations. If  $W \leq V$ , then for any positive integer  $n$ , if  $w_1, w_2, \dots, w_n \in W$  and  $c_1, \dots, c_n \in F$ , then

$$c_1 w_1 + c_2 w_2 + \dots + c_n w_n \in W.$$

**Proposition 19 (Spans Are Subspaces).**

Let  $S \subseteq V$  (a subset) and  $W \leq V$  (a subspace). Then:

1.  $S \subseteq \text{span}(S)$ .
2. If  $S \subseteq W$ , then  $\text{span}(S) \subseteq W$ .
3.  $\text{span}(W) = W$ .

## 2.3 Linear Independence and Dependence

**Definition 20 (Linear Dependence).**

Let  $V$  be a vector space over  $F$  and  $S \subseteq V$ . The set  $S$  is linearly dependent if there exist distinct vectors  $u_1, \dots, u_n \in S$  and scalars  $c_1, \dots, c_n \in F$ , not all zero, such that

$$c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0.$$

Otherwise,  $S$  is linearly independent.

**Definition 21 (Linear Independence).**

Let  $V$  be a vector space over a field  $F$  and let  $S \subseteq V$ . The set  $S$  is said to be linearly independent if whenever distinct vectors  $u_1, \dots, u_n \in S$  and scalars  $c_1, \dots, c_n \in F$  satisfy

$$c_1u_1 + c_2u_2 + \dots + c_nu_n = 0,$$

it follows that  $c_1 = c_2 = \dots = c_n = 0$ .

**Proposition 22 (Dependency Special Cases).**

1. The empty set  $\emptyset$  is linearly independent.
2. Let  $S \subseteq V$ . If  $\vec{0} \in S$ , then  $S$  is dependent (since  $1 \cdot \vec{0} = \vec{0}$  provides a nontrivial dependence).
3. Let  $u \in V$ . Then  $\{u\}$  is independent if and only if  $u \neq \vec{0}$ . Equivalently,  $\{u\}$  is dependent if and only if  $u = \vec{0}$ .
4. Let  $A \subseteq B \subseteq V$ . Then:
  - (a) If  $A$  is dependent, then  $B$  is also dependent.
  - (b) If  $B$  is independent, then  $A$  is also independent.

**Lemma 23 (Extending a Linearly Independent Set).**

Let  $S \subseteq V$  be a linearly independent set, and let  $w \in V$  with  $w \notin S$ . Then  $S \cup \{w\}$  is independent if and only if  $w \notin \text{span}(S)$ . (Adding a vector already in the span of  $S$  makes the set dependent.)

## 2.4 Basis and Dimension

**Definition 24 (Basis).**

Let  $V$  be a vector space over  $F$  and let  $W \leq V$ . A subset  $\beta \subseteq W$  is called a basis of  $W$  if

1.  $\text{span}(\beta) = W$ ,
2.  $\beta$  is linearly independent.

**Theorem 25 ("Bases Exist").**

Let  $W \leq V$ , and suppose  $W = \text{span}(S)$  for some finite set  $S$ . Then there exists a subset  $\beta \subseteq S$  such that  $\beta$  is a basis of  $W$ . (Any finite spanning set can be reduced to a basis.)

**Theorem 26 ("All Bases Have the Same Size").**

Let  $W = \text{span}(S)$  with  $S$  finite. Then  $W$  has a finite basis, and all bases of  $W$  have the same cardinality. This common number is called  $\dim(W)$ .

**Definition 27 (Dimension).**

A vector space  $V$  (or subspace of one) is called finite-dimensional if it admits a finite basis. The dimension of  $V$ , written  $\dim(V)$ , is the number of vectors in any basis of  $V$ . If no finite basis exists, then  $V$  is infinite-dimensional.

**Theorem 28.**

Every vector space (even one without a finite spanning set) has a basis.

**Proposition 29.**

If we find the general solution to  $A\vec{x} = \vec{0}$  as

$$\vec{x} = t_1 \vec{v}_1 + t_2 \vec{v}_2 + \cdots + t_r \vec{v}_r,$$

where  $t_1, \dots, t_r$  are the free variables, then  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$  forms a basis for  $\text{null}(A)$ .

**Theorem 30.**

Suppose  $\dim(W) = n$  is finite, and let  $S \subseteq W$ .

1. If  $W = \text{span}(S)$ , then  $|S| \geq n$ , and there is a subset  $\beta \subseteq S$  such that  $\beta$  is a basis of  $W$ . (Any spanning set can be “shrunk” to a basis.)
2. If  $S$  is independent, then  $|S| \leq n$ , and there is a basis  $\beta$  of  $W$  such that  $S \subseteq \beta$ . (Any independent set can be “extended” to a basis.)
3. Let  $|S| = n$ . If and only if  $W = \text{span}(S)$ , then  $S$  is independent.

## 2.5 Subspaces

**Proposition 31.**

Let  $W \leq V$ , where  $\dim(V) = n$  is finite.

1.  $\dim(W) \leq n$ .
2.  $\dim(W) = n \iff W = V$ .

**Definition 32 (Column Space).**

Let  $A \in M_{m \times n}(F)$ . The column space of  $A$ , denoted  $\text{col}(A)$ , is the subspace of  $F^m$  spanned by the columns of  $A$ :

$$\text{col}(A) = \text{span}\{\text{columns of } A\} \leq F^m.$$

**Theorem 33.**

Let  $A \in M_{m \times n}(F)$ , and let  $R$  be the row-reduced echelon form of  $A$ .

1. A basis for  $\text{row}(A)$  is given by the nonzero rows of  $R$ .
2. The columns of  $A$  that correspond to the leading entries (pivots) in  $R$  form a basis for  $\text{col}(A)$ .

**Theorem 34.**

Let  $U \leq V$  and  $W \leq V$  be subspaces of  $V$ . Then

$$U \cap W = \{v \in V : v \in U \text{ and } v \in W\}$$

is a subspace of  $V$ .

**Definition 35 (Sum of Subspaces).**

Let  $V$  be a vector space over a field  $F$  and let  $U, W \leq V$ . The sum of  $U$  and  $W$  is the subspace

$$U + W = \{u + w \mid u \in U, w \in W\} \leq V.$$

**Proposition 36.**

Let  $U, W$  be subspaces of  $V$ . Then:

1.  $U + W = \text{span}(U \cup W)$ .
2.  $U \leq U + W$  and  $W \leq U + W$ .

**Definition 37 (Direct Sum).**

Suppose  $U, W \leq V$  are subspaces such that every  $v \in V$  can be written uniquely as

$$v = u + w \quad \text{with } u \in U, w \in W.$$

Then  $V$  is called the direct sum of  $U$  and  $W$ , and we write

$$V = U \oplus W.$$

**Proposition 38.**

Let  $U, W$  be subspaces of  $V$ . Then

$$V = U \oplus W \iff (V = U + W) \text{ and } (U \cap W = \{\vec{0}\}).$$

**Theorem 39 (Inclusion-Exclusion Theorem).**

Let  $U, W$  be finite-dimensional subspaces of  $V$ . Then

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

## 2.6 Lagrange Interpolation

**Definition 40 (Lagrange Polynomials).**

Let  $a_0, \dots, a_n \in \mathbb{R}$  be distinct. For each  $i = 0, 1, \dots, n$ , the Lagrange polynomial  $\ell_i(x)$  is defined by

$$\ell_i(x) = \prod_{\substack{0 \leq j \leq n \\ j \neq i}} \frac{x - a_j}{a_i - a_j}.$$

Each  $\ell_i(x)$  has degree  $n$  and satisfies  $\ell_i(a_j) = \delta_{ij}$ .

**Definition 41 (Kronecker Delta).**

The Kronecker delta  $\delta_{ij}$  is defined for any indices  $i, j$  by

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

**Proposition 42.**

1. If  $l_i(a_j) = \delta_{i,j}$ , then  $\delta_{i,j} = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$  (Kronecker delta).
2. The polynomials  $l_0(x), l_1(x), \dots, l_n(x)$  form a basis of  $P_n(\mathbb{R})$ .

## 3 Linear Transformations

### 3.1 Definition and Basic Properties

**Definition 43 (Linear Transformation).**

Let  $U$  and  $V$  be vector spaces over a field  $F$ , and let  $T: U \rightarrow V$  be a function. If for all  $u_1, u_2 \in U$  and all  $c \in F$ ,

$$T(u_1 + u_2) = T(u_1) + T(u_2), \quad T(cu) = cT(u),$$

then  $T$  is called a linear transformation.

**Proposition 44 (Properties of Linear Transformations).**

Let  $T: U \rightarrow V$  be a linear transformation. Then:

1.  $T(\vec{0}) = \vec{0}$ .
2. For all  $u_1, \dots, u_n \in U$  and  $c_1, \dots, c_n \in F$ ,

$$T\left(\sum_{i=1}^n c_i u_i\right) = \sum_{i=1}^n c_i T(u_i).$$

**Definition 45 (Matrix-Induced Linear Map).**

For any  $A \in M_{m \times n}(F)$ , define the map

$$L_A: F^n \longrightarrow F^m, \quad L_A(v) = Av,$$

for all  $v \in F^n$ . One checks that  $L_A$  is linear.

**Proposition 46.**

$L_A$  is a linear transformation.

## 3.2 Kernel and Image

### Definition 47 (Kernel and Image).

Let  $T: U \rightarrow V$  be a linear transformation. The kernel of  $T$  is

$$\ker(T) = \{u \in U \mid T(u) = 0\} \subseteq U,$$

and the image of  $T$  is

$$\text{Im}(T) = \{v \in V \mid v = T(u) \text{ for some } u \in U\} \subseteq V.$$

### Proposition 48.

Let  $T: U \rightarrow V$  be a linear transformation. Then:

1.  $\ker(T) \leq U$  (i.e.,  $\ker(T)$  is a subspace of  $U$ ).
2.  $\text{Im}(T) \leq V$  (i.e.,  $\text{Im}(T)$  is a subspace of  $V$ ).

### Definition 49 (Rank and Nullity).

Let  $T: U \rightarrow V$  be a linear transformation between finite-dimensional spaces. Then

$$\text{rank}(T) = \dim(\text{Im}(T)), \quad \text{nullity}(T) = \dim(\ker(T)).$$

### Proposition 50 (Spanning Set of $\text{Im}(T)$ ).

$T: U \rightarrow V$  linear,  $U = \text{span}(\alpha)$ .

Denote  $T(\alpha) = \{T(u) \mid u \in \alpha\}$ . Then  $T(\alpha)$  spans  $\text{Im}(T)$ .

### Theorem 51 (Rank-Nullity Theorem).

$T: U \rightarrow V$  linear transformation, with  $\dim(U) = n$  finite. Then

$$\text{rank}(T) + \text{nullity}(T) = \dim(U).$$

## 3.3 Injective, Surjective

### Definition 52 (Injective, Surjective, Bijective).

Let  $f: X \rightarrow Y$  be a function between sets  $X$  and  $Y$ .

- $f$  is injective (one-to-one) if

$$\forall x_1, x_2 \in X, x_1 \neq x_2 \implies f(x_1) \neq f(x_2),$$

equivalently  $f(x_1) = f(x_2) \implies x_1 = x_2$ .

- $f$  is surjective (onto) if

$$\forall y \in Y \exists x \in X \text{ with } f(x) = y,$$

equivalently  $\text{Im}(f) = Y$ .

- $f$  is bijective if it is both injective and surjective.

**Proposition 53.**

$T : U \rightarrow V$  linear,  $U, V$  finite dimensional.

1.  $T$  injective  $\iff \ker(T) = \{\vec{0}\}$  ( $\implies \text{nullity}(T) = 0$ ).
2.  $T$  surjective  $\iff \text{Im}(T) = V$  ( $\implies \text{rank}(T) = \dim(V)$ ).
3. If  $\dim(U) = \dim(V)$ , then  $T$  injective  $\iff T$  surjective.
4. If  $\dim(U) > \dim(V)$ ,  $T$  is not injective. If  $\dim(U) < \dim(V)$ ,  $T$  is not surjective.

### 3.4 Isomorphism and Coordinates

**Definition 54 (Isomorphism of Vector Spaces).**

Let  $T : U \rightarrow V$  be a linear transformation between vector spaces over the same field.

- If  $T$  is bijective, then  $T$  is called an isomorphism.
- When such an isomorphism  $T$  exists, we say  $U$  is isomorphic to  $V$  and write

$$U \cong V.$$

**Proposition 55.**

Let  $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$  be a basis of  $V$ . Then every  $\vec{v} \in V$  has a unique expression

$$\vec{v} = \sum_{i=1}^n c_i \vec{v}_i$$

as a linear combination of basis elements. The vector  $(c_1, c_2, \dots, c_n) \in F^n$  of coefficients is called the coordinate vector of  $\vec{v}$  relative to  $\beta$ , denoted  $[\vec{v}]_\beta = (c_1, \dots, c_n)$ .

**Theorem 56.**

Let  $V$  and  $W$  be vector spaces, let  $\alpha = \{v_1, \dots, v_n\}$  be a basis of  $V$ , and let  $w_1, \dots, w_n \in W$  be arbitrary. Then there is a unique linear transformation  $T : V \rightarrow W$  such that

$$T(v_i) = w_i \quad (i = 1, \dots, n).$$

Moreover, if  $v = \sum_{i=1}^n c_i v_i$ , then

$$T(v) = \sum_{i=1}^n c_i w_i.$$

**Corollary 57.**

If  $T, S : V \rightarrow W$  are linear and  $\alpha = \{v_1, \dots, v_n\}$  is a basis of  $V$  with

$$T(v_i) = S(v_i) \quad \text{for } i = 1, \dots, n,$$

then  $T = S$ .

**Theorem 58.**

$V$  vector space,  $\dim V = n$ , finite,  $\beta$  basis. The function which computes coordinates

$$[-]_{\beta} : V \rightarrow F^n$$

is an isomorphism. Hence  $V \cong F^n$  ( $n = \dim V$ ).

**Proposition 59 (Composition and Inverses of Lin. Transforms).**

Let  $S : U \rightarrow V$ ,  $T : V \rightarrow W$  be linear.

1. The composition  $T \circ S : U \rightarrow W$  is linear.
2. If  $S, T$  both isomorphisms,  $T \circ S$  also is an isomorphism.
3. If  $T$  is an isomorphism, there is an inverse  $T^{-1} : W \rightarrow V$ , and  $T^{-1}$  is an isomorphism.

**3.5 Matrix of a Linear Transformation****Theorem 60 (“Only Dimensions Matter”).**

Let  $U, V$  be finite-dimensional vector spaces over  $F$ . Then

$$U \cong V \quad (\iff) \quad \dim(U) = \dim(V).$$

**Definition 61 (Matrix of a Linear Transformation).**

Let  $T : U \rightarrow V$  be a linear transformation, and let  $\alpha = \{u_1, \dots, u_n\}$  and  $\beta = \{v_1, \dots, v_m\}$  be ordered bases of  $U$  and  $V$ , respectively. The matrix of  $T$  relative to  $\alpha$  and  $\beta$  is the  $m \times n$  matrix

$$[T]_{\alpha}^{\beta} = ([T(u_1)]_{\beta} \mid \dots \mid [T(u_n)]_{\beta}),$$

whose  $i$ th column is the coordinate vector of  $T(u_i)$  in the basis  $\beta$ .

**Theorem 62 (“ $[T]_{\alpha}^{\beta}$  Computes  $T$  in Coordinates”).**

Let  $T : U \rightarrow V$  be linear,  $\alpha, \beta$  bases of  $U, V$ . Then for all  $u \in U$ ,

$$[T]_{\alpha}^{\beta} [u]_{\alpha} = [T(u)]_{\beta}.$$

**Proposition 63 (ker and Im in Coordinates).**

Let  $T : U \rightarrow V$ , and let  $\alpha, \beta$  be bases of  $U$  and  $V$ . Define  $A = [T]_{\alpha}^{\beta}$ .

1.  $\ker(T)$  corresponds, via  $\alpha$ -coordinates, to  $\text{null}(A)$  (i.e. the solution set of  $A\vec{x} = \vec{0}$ ).  
In particular,

$$\text{nullity}(T) = \text{nullity}(A) = \#\{\text{free variables}\}.$$

2.  $\text{Im}(T)$  corresponds, via  $\beta$ -coordinates, to  $\text{Im}(L_A) = \text{col}(A)$ . Hence

$$\text{rank}(T) = \text{rank}(A).$$

**Definition 64 (Linear Operator).**

Let  $T: V \rightarrow V$  be a linear transformation on a vector space  $V$  (the domain and codomain are the same vector space). Such a  $T$  is called a linear operator. If  $\dim(V) = n$ , then its matrix relative to any basis  $\alpha$  of  $V$  is an  $n \times n$  matrix, commonly denoted

$$[T]_{\alpha}^{\alpha} = [T]_{\alpha},$$

or simply  $[T]$  when the basis is understood.

**Proposition 65.**

Let  $T: V \rightarrow V$  be a linear operator, and let  $\alpha$  be a basis of  $V$  with  $n = \dim(V)$ . Then

$$T \text{ is invertible} \iff [T]_{\alpha} \text{ is invertible.}$$

Moreover, if  $T$  is invertible, then

$$[T^{-1}]_{\alpha} = ([T]_{\alpha})^{-1}.$$

### 3.6 Change of Basis

**Definition 66 (Change-of-Coordinates Matrix).**

Let  $V$  be an  $n$ -dimensional vector space over a field  $F$ , and let  $\alpha$  and  $\beta$  be two ordered bases of  $V$ . The change-of-coordinates matrix from  $\alpha$  to  $\beta$  is the  $n \times n$  matrix

$$Q_{\alpha}^{\beta} = [I]_{\alpha}^{\beta},$$

where  $I: V \rightarrow V$  is the identity. Equivalently, the  $j$ th column of  $Q_{\alpha}^{\beta}$  is the coordinate vector of the  $j$ th basis vector of  $\alpha$  expressed in the basis  $\beta$ .

**Proposition 67.**

1. For every  $u \in V$ ,

$$Q_{\alpha}^{\beta} [u]_{\alpha} = [u]_{\beta}.$$

2.  $Q_{\alpha}^{\beta}$  is invertible and

$$(Q_{\alpha}^{\beta})^{-1} = Q_{\beta}^{\alpha}.$$

**Theorem 68 (Similarity Transformation).**

Let  $T : V \rightarrow V$  be a linear operator and let  $\alpha, \beta$  be two bases of  $V$ . Then

$$Q_\alpha^\beta [T]_\alpha Q_\beta^\alpha = [T]_\beta.$$

**Definition 69 (Similar Matrices).**

Let  $A, B \in M_{n \times n}(F)$ . We say  $A$  and  $B$  are similar if there exists an invertible matrix  $Q \in M_{n \times n}(F)$  such that

$$Q^{-1} A Q = B.$$

**Proposition 70.**

Let  $Q \in M_{n \times n}(F)$  be any invertible matrix and let  $\alpha$  be a basis of  $V$  (so  $n = \dim V$ ). Then there exists a basis  $\beta$  of  $V$  such that

$$Q = Q_\alpha^\beta.$$

**Proposition 71.**

Let  $T : V \rightarrow V$  be a linear operator, let  $\alpha$  be a basis of  $V$ , and let  $B \in M_{n \times n}(F)$  be any matrix. Then

$$[T]_\alpha \text{ is similar to } B \iff \exists \text{ a basis } \beta \text{ of } V \text{ with } B = [T]_\beta.$$

**Theorem 72.**

If  $\dim V = n$  and  $\dim W = m$ , then

$$\mathcal{L}(V, W) \cong M_{m \times n}(F).$$

Moreover, given a basis  $\alpha$  of  $V$  and a basis  $\beta$  of  $W$ , the map

$$\varphi : \mathcal{L}(V, W) \longrightarrow M_{m \times n}(F), \quad \varphi(T) = [T]_\alpha^\beta$$

is a vector-space isomorphism.

## 4 Inner Product Spaces

### 4.1 Definition and Main Examples

**Definition 73 (Inner Product).**

Let  $V$  be a vector space over  $\mathbb{F}$  (where  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ ). An inner product on  $V$  is a function

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{F}$$

satisfying for all  $u, v, w \in V$  and all scalars  $c \in \mathbb{F}$ :

$$1. \langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle, \quad \langle u, cv \rangle = c \langle u, v \rangle. \quad (\text{linearity}).$$

2.  $\langle u, v \rangle = \overline{\langle v, u \rangle}$  (conjugate symmetry).
3.  $\langle u, u \rangle > 0$  for all  $u \neq 0$  (positive definiteness).

**Proposition 74 (Inner Product Properties).**

Let  $V$  be an inner-product space over  $\mathbb{F}$  and let  $u, v, w \in V$ ,  $c \in \mathbb{F}$ . Then:

1. (Conjugate-linearity in the second slot)

$$\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle, \quad \langle u, cv \rangle = \bar{c} \langle u, v \rangle.$$

2.  $\langle u, 0 \rangle = 0$  and  $\langle 0, u \rangle = 0$ .
3.  $\langle u, u \rangle = 0$  if and only if  $u = 0$ .
4. If  $\langle u, w \rangle = \langle v, w \rangle$  for all  $w \in V$ , then  $u = v$ .

**Definition 75 (Conjugate and Adjoint of a Matrix).**

Let  $A = (a_{ij}) \in M_{m \times n}(\mathbb{F})$ .

- The conjugate of  $A$  is the entrywise-conjugated matrix

$$\bar{A} = (\bar{a}_{ij}) \in M_{m \times n}(\mathbb{F}).$$

- If  $A$  is square ( $n = m$ ), the adjoint (or Hermitian transpose) of  $A$  is

$$A^* = \bar{A}^T = (\bar{A})^T \in M_{n \times n}(\mathbb{F}),$$

characterized by  $\langle Au, v \rangle = \langle u, A^*v \rangle$  for all  $u, v$ .

## 4.2 Norm and Angle

**Definition 76 (Norm).**

Let  $V$  be an inner-product space with inner product  $\langle \cdot, \cdot \rangle$ . The norm (or length) of a vector  $u \in V$  is

$$\|u\| = \sqrt{\langle u, u \rangle}.$$

**Proposition 77.**

Let  $V$  be an inner-product space,  $u \in V$ , and  $c \in \mathbb{F}$ . Then

$$\|cu\| = |c| \|u\|.$$

**Theorem 78 (Cauchy–Schwarz Inequality).**

Let  $V$  be an inner-product space and  $u, v \in V$ . Then

1.  $|\langle u, v \rangle| \leq \|u\| \|v\|$ .

2. Equality holds,  $|\langle u, v \rangle| = \|u\| \|v\|$ , if and only if  $u = cv$  (equivalently  $v = cu$ ) for some scalar  $c \in \mathbb{F}$ .

**Proposition 79 (Triangle Inequality).**

In any inner-product space  $V$ , for all  $u, v \in V$ ,

$$\|u + v\| \leq \|u\| + \|v\|.$$

**Definition 80 (Angle).**

Let  $V$  be a real inner-product space and let  $u, v \in V$  be nonzero. By the Cauchy–Schwarz inequality, the quotient

$$\frac{\langle u, v \rangle}{\|u\| \|v\|}$$

lies in  $[-1, 1]$ , so there is a unique  $\theta \in [0, \pi]$  with

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\| \|v\|}.$$

This  $\theta$  is called the angle between  $u$  and  $v$ .

### 4.3 Orthogonal Sets and Complements

**Definition 81 (Orthogonal and Orthonormal Sets).**

Let  $V$  be an inner-product space.

1. Two vectors  $u, v \in V$  are called orthogonal if  $\langle u, v \rangle = 0$ . (In particular, the zero-vector is orthogonal to every vector.)
2. A subset  $X \subseteq V$  is orthogonal if every pair of distinct vectors in  $X$  is orthogonal and  $0 \notin X$ .
3. If, in addition,  $\|u\| = 1$  for all  $u \in X$ , then  $X$  is orthonormal.
4. An orthonormal subset that is also a basis of  $V$  is called an orthonormal basis (ONB).

**Proposition 82.**

Let  $V$  be an inner-product space and let  $X \subseteq V$  be an orthogonal set. Then  $X$  is linearly independent.

**Theorem 83 (Fourier Coefficients).**

Let  $\alpha = \{v_1, \dots, v_n\}$  be an orthogonal basis of the inner-product space  $V$ . Then for every  $u \in V$ ,

$$u = \sum_{i=1}^n \frac{\langle u, v_i \rangle}{\langle v_i, v_i \rangle} v_i,$$

and the coordinate vector  $[u]_\alpha$  is

$$[u]_\alpha = \begin{pmatrix} \frac{\langle u, v_1 \rangle}{\langle v_1, v_1 \rangle} \\ \vdots \\ \frac{\langle u, v_n \rangle}{\langle v_n, v_n \rangle} \end{pmatrix}.$$

If  $\alpha$  is orthonormal, then this simplifies to

$$u = \sum_{i=1}^n \langle u, v_i \rangle v_i.$$

**Definition 84 (Orthogonal Complement).**

Let  $V$  be an inner-product space and  $X \subseteq V$ . The orthogonal complement of  $X$  is

$$X^\perp = \{v \in V \mid \langle v, x \rangle = 0 \text{ for all } x \in X\},$$

which is a subspace of  $V$ .

**Proposition 85.**

Let  $W \leq V$ . Then:

1.  $W^\perp$  is a subspace of  $V$ .
2. If  $\alpha = \{w_1, \dots, w_k\}$  is a basis of  $W$ , then for any  $u \in V$ ,

$$u \in W^\perp \iff \langle u, w_i \rangle = 0 \quad (i = 1, \dots, k).$$

3.  $W \cap W^\perp = \{0\}$ .

## 4.4 Orthogonal Projection and Gram-Schmidt Algorithm

**Theorem 86 (Orthogonal Projection).**

Let  $V$  be an inner-product space and  $W \leq V$  a finite-dimensional subspace. Then:

1. For each  $u \in V$  there are unique  $w \in W$  and  $w' \in W^\perp$  such that  $u = w + w'$ . Hence  $V = W \oplus W^\perp$ .
2. If  $\{u_1, \dots, u_k\}$  is an orthogonal basis of  $W$ , then the projection  $w \in W$  of any  $u \in V$  is

$$w = \sum_{i=1}^k \frac{\langle u, u_i \rangle}{\langle u_i, u_i \rangle} u_i.$$

3. If  $V$  is finite-dimensional, then  $\dim V = \dim W + \dim W^\perp$ .

**Lemma 87 (Pythagoras' Theorem).**

Let  $V$  be an inner-product space and let  $u, v \in V$  be orthogonal. Then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

**Theorem 88 (Projection is Closest Vector in  $W$ ).**

Let  $V$  be an inner-product space,  $W \leq V$  finite-dimensional, and  $u \in V$ . Write the unique decomposition

$$u = w + w', \quad w \in W, \quad w' \in W^\perp,$$

so that  $w = P_W(u)$ . Then  $w$  is the closest vector in  $W$  to  $u$ , in the sense that for every  $z \in W$ ,

$$\|u - w\| \leq \|u - z\|.$$

**Theorem 89 (Gram–Schmidt Orthogonalization).**

With the notation of the Gram–Schmidt process applied to a basis  $\{u_1, \dots, u_n\} \subset V$ , one defines vectors  $v_1, \dots, v_n$ . Then:

1. For each  $k = 1, \dots, n$ ,

$$\text{span}\{v_1, \dots, v_k\} = \text{span}\{u_1, \dots, u_k\},$$

hence  $\{v_1, \dots, v_n\}$  spans the same subspace  $W$ .

2. For each  $k$ , the set  $\{v_1, \dots, v_k\}$  is orthogonal in  $W_k = \text{span}\{u_1, \dots, u_k\}$ , and in particular  $\{v_1, \dots, v_n\}$  is an orthogonal basis of  $W$ .

## 4.5 Inner Product Defined by a Matrix

**Definition 90 (Positive Definite Matrix).**

Let  $A \in M_{n \times n}(\mathbb{R})$  be symmetric.  $A$  is called positive definite if

$$u^T A u > 0 \quad \text{for all nonzero } u \in \mathbb{R}^n.$$

**Proposition 91.**

Let  $A \in M_{n \times n}(\mathbb{R})$  be symmetric and positive definite. Then

$$\langle u, v \rangle_A = u^T A v$$

defines an inner product on  $\mathbb{R}^n$ .

## 5 Eigenvalues and Diagonalization

### 5.1 Eigenvalues and Eigenvectors

**Definition 92 (Eigenvector and Eigenvalue).**

Let  $T: V \rightarrow V$  be a linear operator on a vector space  $V$ . A nonzero vector  $u \in V$  is an eigenvector of  $T$  if there exists a scalar  $\lambda \in F$  such that

$$T(u) = \lambda u.$$

The scalar  $\lambda$  is called the corresponding eigenvalue.

**Definition 93 (Eigenspace).**

If  $\lambda$  is an eigenvalue of  $T: V \rightarrow V$ , the eigenspace of  $\lambda$  is the subspace

$$E_\lambda = \{u \in V \mid T(u) = \lambda u\}.$$

**Definition 94 (Characteristic Polynomial).**

Let  $T: V \rightarrow V$  be a linear operator on an  $n$ -dimensional vector space  $V$ . The characteristic polynomial of  $T$  is

$$C_T(x) = \det([T]_\alpha - xI),$$

where  $\alpha$  is any basis of  $V$  and  $[T]_\alpha$  is the matrix of  $T$  relative to  $\alpha$ .

**Proposition 95 (Characteristic Polynomial).**

Let  $T: V \rightarrow V$  be a linear operator on an  $n$ -dimensional space  $V$ . Its characteristic polynomial  $C_T(x)$  satisfies:

1.  $C_T(x)$  is independent of the choice of basis.
2.  $\deg C_T = n$ .
3. A scalar  $\lambda$  is an eigenvalue of  $T$  if and only if  $C_T(\lambda) = 0$ .
4. The eigenspace  $E_\lambda = \ker(T - \lambda I)$  is a subspace of  $V$ .

### 5.2 Diagonalization

**Definition 96 (Diagonalizable Operator).**

Let  $T: V \rightarrow V$  be a linear operator on a finite-dimensional vector space  $V$ . We say  $T$  is diagonalizable if there exists a basis  $\alpha$  of  $V$  consisting entirely of eigenvectors of  $T$ ; equivalently, in that basis the matrix of  $T$  is diagonal.

**Proposition 97 (Diagonalizability Criterion).**

Let  $T: V \rightarrow V$  be a linear operator on a finite-dimensional space  $V$ . Then  $T$  is diagonalizable if and only if there exists a basis of  $V$  consisting of eigenvectors of  $T$ ; equivalently, for any basis  $\beta$ , the matrix  $[T]_\beta$  is similar to a diagonal matrix.

**Definition 98 (Diagonalizable Matrix).**

Let  $A \in M_{n \times n}(F)$ . We say  $A$  is diagonalizable if there exist an invertible matrix  $Q \in M_{n \times n}(F)$  and a diagonal matrix  $D \in M_{n \times n}(F)$  such that

$$Q^{-1} A Q = D.$$

**5.3 Diagonalizability****Proposition 99.**

Let  $V$  be a finite-dimensional real vector space and  $T : V \rightarrow V$  linear. If the characteristic polynomial  $C_T(x)$  has a nonreal complex root, then  $T$  is not diagonalizable over  $\mathbb{R}$ .

**Lemma 100.**

Let  $T : V \rightarrow V$  be linear and let  $\lambda_1 \neq \lambda_2$  be two distinct eigenvalues. Then

$$E_{\lambda_1} \cap E_{\lambda_2} = \{0\}.$$

**Proposition 101.**

Let  $T : V \rightarrow V$  have distinct eigenvalues  $\lambda_1, \dots, \lambda_m$ . For each  $i$  let  $\beta_i$  be a basis of the eigenspace  $E_{\lambda_i}$ , and set  $\beta = \beta_1 \cup \dots \cup \beta_m$ . Then:

1.  $|\beta| = \sum_i |\beta_i|$ .
2.  $\beta$  is linearly independent.

**Corollary 102.**

If  $T : V \rightarrow V$  is linear on an  $n$ -dimensional space and has  $n$  distinct eigenvalues, then  $T$  is diagonalizable.

**Definition 103 (Geometric and Algebraic Multiplicity).**

Let  $T : V \rightarrow V$  be a linear operator and  $\lambda$  an eigenvalue of  $T$ .

- The geometric multiplicity of  $\lambda$  is  $\dim E_\lambda$ , where  $E_\lambda = \ker(T - \lambda I)$ .
- The algebraic multiplicity of  $\lambda$  is the exponent of the factor  $(x - \lambda)$  in the characteristic polynomial  $C_T(x)$ .

**Theorem 104.**

Let  $T : V \rightarrow V$  be a linear operator on a finite-dimensional space  $V$ , and let  $\lambda$  be any eigenvalue of  $T$ . Then

$$1 \leq (\text{geometric multiplicity of } \lambda) \leq (\text{algebraic multiplicity of } \lambda).$$

**Proposition 105.**

Let  $T : V \rightarrow V$  be a linear operator on an  $n$ -dimensional vector space over a field  $F$ , with characteristic polynomial  $C_T(x)$ .

1. If  $F = \mathbb{R}$  and  $C_T(x)$  has a nonreal root, then  $T$  is not diagonalizable over  $\mathbb{R}$ .
2. If  $F = \mathbb{R}$  and all roots of  $C_T(x)$  lie in  $\mathbb{R}$ , or if  $F = \mathbb{C}$ , then

$$T \text{ is diagonalizable} \iff \forall \lambda, (\text{geom. mult. } \lambda) = (\text{alg. mult. } \lambda).$$

3. When  $T$  is diagonalizable, let  $\lambda_1, \dots, \lambda_k$  be its distinct eigenvalues and  $\beta_i$  a basis of the eigenspace  $E_{\lambda_i}$ . Then

$$\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_k$$

is a basis of  $V$  consisting entirely of eigenvectors of  $T$ .

## 5.4 Orthogonal Diagonalization

**Definition 106 (Symmetric, Orthogonal, Self-Adjoint, Unitary Matrices).**

Let  $A \in M_{n \times n}(F)$ .

- If  $F = \mathbb{R}$ ,  $A$  is symmetric when  $A^T = A$ .
- If  $F = \mathbb{R}$ ,  $A$  is orthogonal when its columns form an orthonormal basis of  $\mathbb{R}^n$  (equivalently  $A^T = A^{-1}$ ).
- If  $F = \mathbb{C}$ ,  $A$  is self-adjoint (Hermitian) when  $A^* = A$ .
- If  $F = \mathbb{C}$ ,  $A$  is unitary when its columns form an orthonormal basis of  $\mathbb{C}^n$  (equivalently  $A^* = A^{-1}$ ).

**Proposition 107 (Orthogonal and Unitary Matrices).**

1. If  $A \in M_n(\mathbb{R})$ , then  $A$  is orthogonal  $\Leftrightarrow A^T = A^{-1}$ .
2. If  $A \in M_n(\mathbb{C})$ , then  $A$  is unitary  $\Leftrightarrow A^* = A^{-1}$ .

**Definition 108 (Orthogonal and Unitary Diagonalizability).**

Let  $A \in M_{n \times n}(F)$ .

- If  $F = \mathbb{R}$ ,  $A$  is orthogonally diagonalizable if there exists an orthogonal matrix  $Q$  and a diagonal  $D$  with

$$Q^T A Q = D \iff A = Q D Q^T.$$

- If  $F = \mathbb{C}$ ,  $A$  is unitarily diagonalizable if there exists a unitary matrix  $Q$  and a diagonal  $D$  with

$$Q^* A Q = D \iff A = Q D Q^*.$$

**Proposition 109.**

1. If  $A \in M_n(\mathbb{R})$  is orthogonally diagonalizable, then  $A$  is symmetric ( $A^T = A$ ).

2. If  $A \in M_n(\mathbb{C})$  is unitarily diagonalizable, then  $A$  is normal ( $A^*A = AA^*$ ).

**Lemma 110.**

1. If  $A \in M_n(\mathbb{R})$  and  $u, v \in \mathbb{R}^n$ , then

$$\langle Au, v \rangle = \langle u, A^T v \rangle, \quad \langle u, Av \rangle = \langle A^T u, v \rangle.$$

2. If  $A \in M_n(\mathbb{C})$  and  $u, v \in \mathbb{C}^n$ , then

$$\langle Au, v \rangle = \langle u, A^* v \rangle, \quad \langle u, Av \rangle = \langle A^* u, v \rangle.$$

**Theorem 111 (Reality of Eigenvalues).**

If  $A \in M_n(\mathbb{R})$  is symmetric or  $A \in M_n(\mathbb{C})$  is self-adjoint, then all roots of its characteristic polynomial lie in  $\mathbb{R}$ ; equivalently, every eigenvalue of  $A$  is real.

**Theorem 112 (Orthogonality of Eigenspaces).**

Let  $A \in M_n(\mathbb{R})$  be symmetric or  $A \in M_n(\mathbb{C})$  be normal. Then its eigenspaces are mutually orthogonal: if  $u \in E_{\lambda_1}$  and  $w \in E_{\lambda_2}$  with  $\lambda_1 \neq \lambda_2$ , then  $\langle u, w \rangle = 0$ .

**Theorem 113 (Spectral Theorem).**

1. For  $A \in M_n(\mathbb{R})$ :

$$A \text{ orthogonally diagonalizable} \iff A \text{ symmetric } (A^T = A).$$

2. For  $A \in M_n(\mathbb{C})$ :

$$A \text{ unitarily diagonalizable} \iff A \text{ normal } (A^*A = AA^*).$$

**Definition 114 (Definiteness of a Symmetric Matrix).**

Let  $A \in M_{n \times n}(\mathbb{R})$  be symmetric. We say:

- $A$  is positive definite if  $u^T A u > 0$  for all nonzero  $u \in \mathbb{R}^n$ .
- $A$  is positive semidefinite if  $u^T A u \geq 0$  for all  $u \in \mathbb{R}^n$ .
- $A$  is negative definite if  $u^T A u < 0$  for all nonzero  $u$ .
- $A$  is negative semidefinite if  $u^T A u \leq 0$  for all  $u$ .
- Otherwise,  $A$  is called indefinite.

**Theorem 115 (Positive Definiteness).**

Let  $A \in M_n(\mathbb{R})$  be symmetric. Then

$$A \text{ is positive definite} \iff \text{all eigenvalues of } A \text{ are positive.}$$

**Lemma 116.**

Let  $A \in M_{m \times n}(\mathbb{R})$ . Then:

1.  $A^T A \in M_{n \times n}(\mathbb{R})$  and  $AA^T \in M_{m \times m}(\mathbb{R})$  are symmetric and positive semi-definite.
2.  $\text{rank}(A^T A) = \text{rank}(A) = \text{rank}(AA^T)$ .



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