1. Determine the general solution of the PDE

$$x(y-u)u_x + y(u-x)u_y = (x-y)u$$

and then determine that solution passing through the line $x = y = u$.

2. The neutron density in an absorbing medium obeys the PDE

$$n_t = n_{xx} - n\,, \qquad -\infty < x < \infty.$$

At $t = 0$, a burst of neutrons is produced such that $n(x,0) = f(x)$.

(a) Find $n(x,t)$ for $t > 0$ using the convolution theorem.

(b) What is $n(x,t)$ if $f(x) = \delta(x - x_0)$, the Dirac delta function?

3. Transform the following PDE to canonical form and then find its general solution:

$$y^2 u_{xx} - 2y u_{xy} + u_{yy} - u_x = 6y.$$

(Helpful suggestion: Solve the transformed equation to obtain the homogeneous solution, but the simplest particular integral can be obtained from the original PDE).

4. If the temperature of a spherical surface $r = a$ is maintained at $u(a,\phi) = 1 + \cos\phi$, determine the steady-state temperature *outside* the sphere if u tends to zero as $r \to \infty$. When there is axial symmetry, Laplace's equation is:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\frac{\partial u}{\partial \phi}\right) = 0.$$

5. Consider the propagation of a nonlinear wave as described by

$$u_t + uu_x + \gamma u = 0, \quad u(x,0) = f(x),$$

where γ is a positive constant. Make a rough sketch of the characteristics on an xt diagram and show that wave breaking will occur eventually only if $f'(x) < -\gamma$ somewhere.

6. The equations of one-dimensional gas dynamics for isentropic flows are the following:

$$\begin{aligned} \rho\, u_x + u\, \rho_x + \rho_t &= 0 \\ u\, u_x + \frac{c^2}{\rho}\, \rho_x + u_t &= 0, \end{aligned}$$

where u and ρ are the density and velocity and $c^2(\rho)$ is the sound speed squared. Find the characteristic directions for the system and the ODEs that apply along the characteristics. Finally, show that the Riemann invariants are

$$u \pm \int \frac{c(\rho)}{\rho}\, d\rho = \text{ constant.}$$

7. For what value(s) of the constant α do solutions exist for the nonhomogeneous boundary value problem

$$x^2\phi'' + x\phi' + 4\phi = \log x^2 + \alpha \sin(\log x^2), \;\; \phi(1) = \phi(e^\pi) = 0?$$

(Recall that $x^{\delta + i\mu} = x^\delta\{\cos(\mu \log x) + i \sin(\mu \log x)\}$.)

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-586A

APPLIED PARTIAL DIFFERENTIAL EQUATIONS

Examiner: Professor S.A. Maslowe	Date: Monday, December 13, 1999
Associate Examiner: L.J. Campbell	Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

THIS IS AN OPEN BOOK EXAMINATION.

This exam comprises the cover and two pages of questions.