Student Name:
Student Id#:

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 579

Numerical Differential Equations

Examiner: Professor A.R. Humphries — Date: April 17, 2009
Associate Examiner: Professor G. Schmidt — Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

1. All questions carry equal weight.
2. Answer 6 or more questions; credit will be given for the best 6 answers.
3. Answer questions in the exam book provided. Start each answer on a new page.
4. This is a closed book exam.
5. Notes and textbooks are not permitted.
6. Calculators are permitted.
7. Translation dictionaries (English-French) are permitted.

This exam comprises of the cover page, and 2 pages of 8 questions.

1. Define the terms *absolute stability* and *A-stability* as applied to a Runge-Kutta method used for solving initial value problems for ordinary differential equations.

   Show that the linear stability function $R(z)$ of an s-stage explicit Runge-Kutta method is a polynomial in z of degree at most s. Hence deduce that

   (a) the maximum order of an explicit s-stage Runge-Kutta method is s,

   (b) no consistent explicit Runge-Kutta method is A-stable.

2. Derive the three stage third order explicit Runge-Kutta method which has $c_2 = c_3$ and $b_2 = b_3$ and state its Butcher Tableau. (You may use the Runge-Kutta order conditions without proof but should state them clearly).

   Find a lower order method which uses the same stages but different weights $\tilde{b}_i$. Hence state an estimate for the local truncation error of the lower order method in terms of the stage values $f(Y_i, t_n + c_i h)$, and give a formula which could be used to update the step-size so that the (estimated) local error is controlled with respect to a tolerance τ.

3. Consider the general $2d$ dimensional Hamiltonian system $\dot{u} = f(u)$ with $u = (x, y)$ and

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x}.$$

   where $x(t) \in \mathbb{R}^d$ and $y(t) \in \mathbb{R}^d$.

   Show that $\dot{H}(u(t)) = 0$ and $\nabla \cdot f(u) = 0$. What is the significance of each of these equalities for solutions of the differential equation (one sentence for each)?

   Apply the forward Euler method with step-size Δt to the 2-dimensional Hamiltonian system defined by $H(x, y) = x^2 + y^2$, where x and y are scalars and show that $H(x_{n+1}, y_{n+1}) = (1 + \Delta t^2) H(x_n, y_n)$.

   Modify the forward Euler method for this problem by applying the *Stabilisation method* $u_{n+1} = u^*_{n+1} - \alpha \nabla H(u^*_{n+1})$ where α solves $H(u^*_{n+1} - \alpha \nabla H(u^*_{n+1})) = H(u_n)$, and derive explicit formulae for x_{n+1} and y_{n+1} in terms of x_n, y_n and Δt for the resulting Hamiltonian conserving method.

4. State and derive the conditions for a linear multistep method to be of order p for any $p \geqslant 1$.

   For what values of the parameter α is the method

$$u_{n+2} + (\alpha - 1)u_{n+1} - \alpha u_n = \frac{h}{2}\Big[(\alpha + 3)f_{n+1} + (\alpha - 1)f_n\Big]$$

   convergent?

5. State the region of absolute stability $\mathcal{S}$ for a linear multistep method in terms of the roots $\xi_i(z)$ of $\tau(\xi)$ where
$$\tau(\xi) = \rho(\xi) - z\sigma(\xi).$$
Consider the two-step method
$$u_{n+2} - \frac{4}{3}u_{n+1} + \frac{1}{3}u_n = \frac{2}{3}hf(t_{n+2}, u_{n+2}).$$
Define strict zero-stability and show that this method has that property.

Use the root locus method to show that if $z \in \partial S$ (where ∂S denotes the boundary of the region of absolute stability) then
$$z(\theta) = (\cos\theta - 1)^2 + i\sin\theta(2 - \cos\theta),$$
for some $\theta \in [0, 2\pi]$.

Deduce whether or not the method is A-stable.

6. Show that every solution of $-u_{xx} + u = f$, for $x \in [0,1] = \Omega$, $u(0) = u(1) = 0$, is also the solution of a related weak problem (which you should state). Show that the weak problem is equivalent to a minimization problem.

Formulate a Galerkin Finite Element Approximation to this problem using a suitable space V_h of piecewise linear basis functions, showing how the method reduces to a linear algebra problem $Ku = F$ by constructing K and F and giving the meaning of the entries of the unknown vector x.

7. Consider the PDE $u_{xx} + u_x = f(x)$, $x \in [0,1]$, with Dirichlet boundary conditions, $u(0) = u_0$ and $u(1) = u_1$. Define a finite difference discretization of this problem, and show that it has second order truncation error (that is show that $L_h u - f = \mathcal{O}(h^2)$ where $Lu = f$ and $Lu \equiv u_{xx} + u_x$ and $L_h u$ is your FD formula applied to the exact solution).

Show that the discretized problem can be formulated as a linear algebra problem $Ax = b$, by constructing A and b and giving the meaning of the entries of the unknown vector x.

If the left hand boundary condition is replaced by $u_x(0) = a$ for a constant a, how could you modify your scheme so as to still retain second order truncation error?

8. Let $u(x,t)$ be the solution of
$$u_t = u_{xx} + \lambda u, \quad \lambda < 0,$$
for $t \geqslant 0$, $x \in [0,1]$ with $u(0,t) = \alpha$, $u(1,t) = \beta$, and $u(x,0) = u_0(x)$.

Consider a numerical solution defined by the finite difference scheme
$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + \lambda u_j^n.$$
Use Von Neumann stability analysis to show that the method is stable under a suitable condition on Δt and Δx, which you should state.

Define the truncation error of the scheme, and show that under a suitable relation between Δt and Δx, it can be written as $\mathcal{O}(\Delta x^p)$ for suitable p (which you should determine).