McGILL UNIVERSITY

FACULTY OF SCIENCE

IN-DEPARTMENT FINAL EXAMINATION

MATH 574

Dynamical Systems

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Date: Wednesday April 15, 2010
Time: 2:00PM - 5:00PM

INSTRUCTIONS

Answer 5 or 6 questions; credit will be given for the best 5 answers.
Please attempt all questions in the exam booklets provided.
Start each question on a new page.
All questions carry equal weight.
This is a closed book exam. No notes, and text books are permitted.
Non-programmable and non-graphical calculators are permitted.
Dictionaries are not permitted.

This exam of comprises the cover page, and 2 pages of 6 questions.

1. Consider the system of differential equations

$$\dot{x} = x - xy - x(x^2 + y^2), \qquad x(0) = x_0 \in \mathbb{R},$$

$$\dot{y} = y + x^2 - y(x^2 + y^2), \qquad y(0) = y_0 \in \mathbb{R}.$$

   (a) By converting to polar coordinates, or otherwise, show that this system of differential equations defines a dynamical system.

   (b) Sketch the phase portrait of the dynamical system.

   (c) State (without proof) the ω-limit set of each initial condition in the plane. Does the dynamical system have a global attractor, and if it does, what is it?

2. Consider a dynamical system defined by

$$\dot{x} = x(2 - x - y),$$

$$\dot{y} = y(3 - 2x - y),$$

   where $x(t) \geqslant 0$ and $y(t) \geqslant 0$ represent the size of competing animal populations.

   (a) Find all the fixed points of this dynamical system which satisfy $x \geqslant 0$, $y \geqslant 0$, and determine their stability types using the Jacobian matrix.

   (b) Sketch the phase portrait for $x \geqslant 0$, $y \geqslant 0$, clearly indicating the fixed points and direction of flow, and making use of the isoclines (nullclines). Label the *stable* and *unstable* manifolds of any saddle points.

   (c) State without proof the ω-limit sets for all initial conditions. What does this is imply for the animal populations?

3. Consider the system of differential equations

$$\dot{x} = -2x + y, \qquad x(0) = x_0 \in \mathbb{R},$$

$$\dot{y} = \mu y - x^3, \qquad y(0) = y_0 \in \mathbb{R},$$

   where $\mu \in \mathbb{R}$ is a bifurcation parameter.

   (a) Find the eigenvalues and eigenvectors of the Jacobian of f at $(x, y) = (0, 0)$ for all values of $\mu \in \mathbb{R}$. Hence, determine the value of the parameter $\mu \in \mathbb{R}$ at which the fixed point at $(0, 0)$ is not hyperbolic. At this value of μ state the linear stable, linear unstable and linear centre manifolds of $(0, 0)$.

   (b) Assuming that near the non hyperbolic point found above the extended centre manifold can be written as

$$y = h(x, \mu) = a(\mu) + b(\mu)x + c(\mu)x^2 + d(\mu)x^3 + \mathcal{O}(x^4),$$

   where a, b c and d are functions of μ only. Find two expressions for $\dot{y}$ on the curve $y = h(x, \mu)$ and hence, or otherwise, determine and state the functions $a(\mu)$, $b(\mu)$, and $c(\mu)$. Find a function $f(x, \mu)$ so that $\dot{x} = f(x, \mu) + \mathcal{O}(x^4)$ describes the dynamics on the extended centre manifold, and use $f(x, \mu)$ (ignoring the $\mathcal{O}(x^4)$ terms) to determine which type of bifurcation occurs.

4. Consider the system of differential equations

$$\dot{x} = y, \qquad x(0) = x_0 \in \mathbb{R},$$

$$\dot{y} = -\mu y - x + x^3, \qquad y(0) = y_0 \in \mathbb{R},$$

where $\mu \in \mathbb{R}$ is a bifurcation parameter.

(a) Show that when $\mu = 0$ that the system of equations can be written as a Hamiltonian system.

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x}.$$

Determine the fixed points of the dynamical system in this case, and their linear stability types and sketch a plausible phase portrait.

(b) Consider the fixed point at the origin for general values of the parameter μ, and show that the eigenvalues λ of the Jacoabian matrix satisfy the characteristic equation

$$\lambda^2 + \mu\lambda + 1 = 0.$$

Hence find the value of μ for which this fixed point is not hyperbolic, and show that $Re(\frac{d\lambda}{d\mu}) \neq 0$ at this value of μ. What type of bifurcation is indicated? (Briefly), would you expect the bifurcation to be supercritical, subcritical or degenerate?

5. Consider the dynamical system $\dot{x} = f(x, \mu)$ where $x \in \mathbb{R}$, and $\mu \in \mathbb{R}$ is a bifurcation parameter.

(a) Assuming that $f(0,0) = 0$, state a necessary condition for a bifurcation to occur at $(x, \mu) = (0,0)$.

(b) By expanding $f(x, \mu)$ in a Taylor series and ignoring higher than quadratic terms, show that if $f_\mu(0,0) \neq 0$ and $f_{xx}(0,0) \neq 0$ then fixed points satisfy

$$x(\mu) = \pm\sqrt{-\frac{2\mu f_\mu(0,0)}{f_{xx}(0,0)}} + \mathcal{O}(\mu).$$

What type of bifurcation is this? Which fixed point is stable?

(c) The derivation of the bifurcation in the previous part is non-rigorous since the higher order terms in the Taylor expansion were ignored. However there is a one line proof of the existence of a continuous curve of fixed points in a neighbourhood of $(x, \mu) = (0,0)$. What is it?

6. Consider the map $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 1 - 2x & x \in [0, 1/2) \\ 2 - 2x & x \in [1/2, 1] \end{cases}$$

(a) Show that eventually periodic points are dense in $[0,1]$.

(b) Show that f has periodic points of prime period n for all $n \geqslant 1$, and that periodic points are also dense in $[0,1]$.

(c) Define topological transitivity and show that the map has this property.