Final Examination April 21, 1997 Mathematics 189-560B

The first 3 questions relate to the following Zermelo navigation problem: A boat, situated at the origin a time $t = 0$ is moving with a velocity v of unit magnitude relative to a stream of constant speed $V = 2$. Its target is the unit disc centered at the point $(5, 2)$. (See the figure.)

1. Formulate the problem of finding a point on the disc, that is closest to the origin, using the Euclidean distance. Solve this convex program using the Karush-Kuhn-Tucker conditions.

2. Formulate the problem of finding a constant steering angle θ that brings the boat to the target in shortest time. Show that this problem can be reduced to an unconstrained minimization problem
$$\text{Min } t = t(\theta) \ .$$
Give the function $t = t(\theta)$. Indicate how you would find a global minimum of this function, but do not solve the problem.

3. Formulate the dual problem of finding the most robust steering angle, i.e., the angle for which the crossing time to the target is least sensitive to small perturbations of the target. Show that this problem can be formulated as an unconstrained minimization problem
$$\text{Min } u = u(\theta) \ .$$
Give the function $u = u(\theta)$ but do not solve the problem numerically.

4. Consider the program

$$(\mathrm{P}) \qquad \begin{array}{l} \text{Min } x_1 \\ \text{s.t.} \\ \qquad x_1 \, x_2 = 0 \\ \qquad x_1 - x_2 = 0 \ . \end{array}$$

   (a) Construct the corresponding classic Langrangian function (with the leading coefficient $\lambda_0 = 1$). Show that the Method of Lagrange does not identify the origin $x^* = (0, 0)$ as a candidate for a local optimum of (P). Why not?

(b) State a general second-order optimality condition that can be used in this situation to verify that a feasible point is an isolated local optimum. Using this condition verify that the origin is an isolated local optimum for the program (P).

5. (a) Formulate and then prove a saddle-point condition that characterizes global optimality of a feasible point of a partly linear program

$$\begin{aligned} &\text{Min } f(x,\theta) \\ &\text{s.t.} \\ &\qquad f^i(x,\theta) \le 0,\ i \in P = \{1,\dots,m\}. \end{aligned}$$

Here $f(\cdot,\theta) : R^n \to R$ is convex and $f^i(\cdot,\theta) : R^n \to R,\ i \in P$ are linear for every $\theta \in R^p$.

(b) Identify the above program (P) as a partly linear program. Then use the saddle-point condition from (a) to prove that

$$x_1^* = \theta^* = 0, \quad x_2^* = x^* = 0$$

is a global optimum.

6. Solve the problem

$$\begin{aligned} &\underset{s.t.}{\text{Min}} \int_0^1 (\dot{x}^2 + 2x)dt \\ &x(0) = x(1) = 0 \end{aligned}$$

using the Euler-Lagrange equation.

7. Consider the problem

$$\begin{aligned} &\underset{s.t.}{\text{Min}}\, J(x) = \int_0^1 (\dot{x}^2 + x^3)dt \\ &\qquad x(0) = 4,\ x(1) = 1\ . \end{aligned}$$

Illustrate the Method of Ritz using the following form of an approximate solution:

$$\Phi(t) = 4 - 3t + \alpha(t - t^2)\ .$$

Find an optimal value of α.

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-560B

OPTIMIZATION

Examiner: Professor S. Zlobec | Date: Monday, April 21, 1997
Associate Examiner: Professor N.G.F. Sancho | Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

Calculators are permitted.
Attempt all problems.

This exam comprises the cover and 2 pages of questions.