Student Name:
Student Id#:

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 387

HONOURS NUMERICAL ANALYSIS

Examiner: Professor A.R. Humphries | Date: Monday April 19, 2010
Associate Examiner: Professor Gantumur Tsogtgerel | Time: 2:00 PM - 5:00 PM

INSTRUCTIONS

1. All questions carry equal weight.
2. **Answer 5 or 6 questions; credit will be given for the best 5 answers.**
3. Answer questions in the exam book provided. Start each answer on a new page.
4. This is a closed book exam.
5. Notes and textbooks are not permitted.
6. Non-programmable calculators are permitted.
7. Translation dictionaries (English-French) are permitted.

This exam comprises of the cover page, and 2 pages of 6 questions.

1. (a) State the Banach fixed point theorem, and show that all the conditions of the theorem are satisfied on a neighbourhood of x^* if $g(x)$ is twice continuously differentiable with $g(x^*) = x^*$ and $|g'(x^*)| < 1$.

   (b) What does it mean for the sequence x_n defined by $x_{n+1} = g(x_n)$ to be convergent to x^* of order p? Show that if g is p-times differentiable with $g(x^*) = x^*$ and $g'(x^*) = g''(x^*) = \ldots = g^{(p-1)}(x^*) = 0$ then the sequence x_n so defined is convergent of order p.

   (c) Find m so that Newton's method applied to $f(x) = (x^2 - a)x^m$ converges (at least) cubically to $\sqrt{a}$. State the resulting iteration in the form $x_{n+1} = g(x_n)$ avoiding square roots in the definition of g.

2. Let $f(x)$ be a given function and let $p_n(x)$ interpolate f at (distinct) interpolation points x_i for $i = 0, \ldots, n$.

   (a) Show that there is at most one polynomial $p_n(x)$ of degree at most n which interpolates the given data.

   (b) Write down the Newton form of the interpolating polynomial, and state the recursion formula used to determine the required divided differences.

   (c) Assuming that f is n-times differentiable, show that there exists $\xi \in [x_0, x_n]$ such that

   $$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

   (d) Given that $p_n(t)$ which interpolates at $x_0, x_1, \ldots, x_n$ is known, what is the formula for $p_{n+1}(t)$ the polynomial of degree $n+1$ which interpolates at $x_0, x_1, \ldots, x_n, x$? Hence show that

   $$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i).$$

3. Let $I(f) = \int_{-1}^{1} f(x)dx$ be approximated by the quadrature rule

   $$I_h(f) = \sum_{i=0}^{n} w_i f(x_i) \qquad (4.1)$$

   (a) Show that for any choice of distinct $x_0, x_1, \ldots, x_n$, there is a quadrature formula with degree of accuracy at least n, but that no matter how the distinct $x_0, x_1, \ldots, x_n$ are chosen there is no method with degree of accuracy more than $2n+1$. (*You may freely use any properties of interpolating polynomials needed, provided you state them clearly*).

   (b) Show that, for a suitable choice of the x_i that a quadrature formula of the form (4.1) with degree of accuracy $2n+1$ can be obtained by integrating a Hermite polynomial of degree $2n+1$;

   $$H_{2n+1}(x) = \sum_{i=0}^{n} \big(f(x_i)\varphi_i(x) + f'(x_i)\psi_i(x)\big),$$

   where

   $$\varphi_i(x) = \big(1 - 2(x - x_i)l_i'(x_i)\big)[l_i(x)]^2, \qquad \psi_i(x) = (x - x_i)[l_i(x)]^2$$

   and $l_i(x)$ is the ith fundamental Lagrange polynomial of degree n. (*You may freely use any properties of Legendre polynomials and interpolating polynomials that you need provided you state them clearly*).

   (c) By considering $f(x) = x^i$ for $i = 0, 1, 2, 3$ find w_0, w_1, x_0, x_1 such that $I_h(f) = w_0 f(x_0) + w_1 f(x_1)$ has degree of accuracy 3.

4. (a) Suppose that $f(x)$ is three times continuously differentiable, and show that

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{1}{2}hf''(\xi),$$

where $\xi \in [x_0, x_0+h]$.

(b) Suppose $|f''(x)| \leqslant M$ for all x, and that $h > 0$. If we encounter roundoff errors δ_1, δ_2 in computing $f(x_0+h), f(x_0)$ respectively, and $|\delta_1|, |\delta_2| < \delta$, find an upper bound on the total error in the approximation to $f'(x_0)$. Determine the value of h which minimises this bound, if $\delta = 2 \times 10^{-8}$ and $M = 100$.

(c) Is the error formula given in (a) appropriate for the direct application of Richardson extrapolation? If not, give an appropriate error formula. Apply one step of Richardson extrapolation to the finite difference approximation in (a) to obtain an approximation to $f''(x_0)$ with $\mathcal{O}(h^2)$ error.

5. Consider the initial value problem

$$y' = f(t, y) = \lambda y, \qquad 0 \leqslant t \leqslant T, \quad y(0) = \alpha > 0, \quad \lambda < 0.$$

Suppose you approximate the solution $y(t)$ using the Runge-Kutta method

$$y_0 = \alpha,$$
$$y_{n+1} = y_n + \tfrac{1}{4}hf(t_n, w_n) + \tfrac{3}{4}hf\left(t_n + \tfrac{2}{3}h, y_n + \tfrac{2}{3}hf(t_n, y_n)\right), \quad n = 0, ...N$$

with time-step h.

(a) Show that $y(t_{n+1}) = e^{h\lambda}y(t_n)$,

(b) and that $y_{n+1} = (1 + h\lambda + \frac{(h\lambda)^2}{2})y_n$.

(c) Under what conditions on h does $\lim_{n\to\infty} y_n = 0$?

(d) Show that $0 < y(t_n) < y_n$ for all $n > 0$.

6. Consider the boundary value problem

$$y'' - y' - 2y = \cos x, \qquad 0 \leqslant x \leqslant \frac{\pi}{2}, \quad y(0) = -0.3, \quad y(\pi/2) = -0.1.$$

(a) Use the Linear Finite Difference method, with second order approximations to each derivative, to formulate a numerical approximation to this problem as a matrix problem $Au = b$, clearly stating the matrix A and vector b as well as the meaning of the elements of the unknown vector x, with $h = \pi/12$. (*Do not attempt to solve the resulting matrix problem*).

(b) Let τ_i be the truncation error in the finite difference equation at the ith mesh point, defined as the residual when the exact solution $y(x)$ is substituted into the finite difference equation. Show that $\tau_i = \mathcal{O}(h^2)$.

(c) Let $e_i = y(x_i) - u_i$ and show that

$$h^2\tau_i = \left[1 - \frac{h}{2}\right]e_{i+1} - [2 + 2h^2]e_i + \left[1 + \frac{h}{2}\right]e_{i-1},$$

and hence that for $h < 2$ that $|e| := \max_i |e_i| = \mathcal{O}(h^2)$.