Note: In questions 1-4 make certain to explain carefully the Sturm-Liouville aspects of each problem justifying the orthogonality of the eigenfunctions, wherever appropriate.

Please leave all answers in simplest form.

1. You may assume that the general solution of Laplace's equation in spherical co-ordinates, i.e., $\nabla^2\psi(r,\varphi) = 0,\ 0 \le \varphi \le \pi,\ \psi$ finite at $\varphi = 0$ and $\varphi = \pi$, is given by $\psi(r,\varphi) = \sum_{n=0}^{\infty}\left[A_n r^n + \frac{B_n}{r^{(n+1)}}\right] P_n(\cos\varphi)$, where P_n are the Legendre polynomials of order n.

   (a) (5 marks) A sphere of radius "a" centered at the origin is placed in a uniform flow with speed V_0 along the z-axis. Find the velocity potential.

   (b) (8 marks) Find the potential distribution inside a hemisphere if the spherical part is maintained at a potential $f(\cos\phi)$ and the flat part is insulated. Hint: Show that the insulation of the flat face, i.e.

   $$\left[\frac{\partial\psi}{\partial z}\right]_{\phi=\pi/2} = 0 \text{ implies } \left[\frac{\partial\psi}{\partial\phi}\right]_{\phi=\pi/2} = 0.$$

   (c) (4 marks) Consider the special case in (a) of $\psi(a,\phi) = V_0(1 + 2\sin^2\phi)$, with V_0 a constant. Note "a" is the radius of the hemisphere.

2. (14 marks) A cylinder occupies the region $0 \le r \le b,\ 0 \le z \le \pi$. It has temperature $f(r,z)$ at time $t = 0$. For $t > 0$, its end $z = 0$ is insulated, and the remaining two surfaces are held at temperature $0°$. Find the temperature inside the cylinder.

3. (16 marks) A sphere of radius b has its surface cooling freely into a medium at a constant temperature T_0. There is a constant heat generation at the rate Q. The initial temperature is $f(r)$. Find the temperature at any point inside the sphere after time t.
   Hint: The temperature $\psi = \psi(r,t)$ only.

4. (14 marks) Solve

   $$\nabla^2\psi(r,\theta) = 0; \quad 1 < r < e,\ 0 < \theta < \alpha.$$

   (i)$\psi_r(1,\theta) = 0,$ (ii) $\psi_r(e,\theta) = 0,$

   (iii) $\psi(r,0) = 0,$ (iv) $\psi(r,\alpha) = f(r)$

   and interpret physically. Note: $e = 2.718....$

5. (a) (8 marks) Solve

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0; \quad 0<x<\infty,\ 0<y<\infty$$

(i) $\psi(x,0)=0$, (ii) $\lim_{x\to\infty}\psi(x,y)=0$,

(iii) $\left[\frac{\partial\psi}{\partial x}\right]_{x=0} \begin{array}{l} =0,\ y>b \\ =-q/K,\ 0<y<b \end{array}$ where q and K are constants.

Leave your answer as a single integral. Interpret physically.

(b) (5 marks) Show that the magnitude of the heat current through the face $y=0$, i.e. $K\left[\frac{\partial\psi}{\partial y}\right]_{y=0}$, where K is the conductivity, is given by $\frac{q}{\pi}\ln\left[1+\frac{b^2}{x^2}\right]$.

6. (10 marks) By first finding the Green's function solve

$$\nabla^2\psi(x,y)=h(x,y); \quad 0<x<\infty,\ 0<y<\infty$$

(i) $\psi(x,0)=f(x)$, (ii) $\psi(0,y)=s(y)$.

It is NOT necessary to compute derivatives of the Green's function.

Explain under what assumption(s) is your solution valid.

7. (16 marks) Obtain the Green's function and then solve

$$\alpha^2\frac{\partial^2\psi}{\partial x^2}-\frac{\partial\psi}{\partial t}=h(x,t); \quad 0<x<\infty,\ t>0$$

(i) $\psi(x,0)=f(x)$, (ii) $\psi_x(0,t)=s(t)$.

Good Luck!

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-375A

DIFFERENTIAL EQUATIONS

Examiner: Professor C. Roth	Date: Thursday, December 16, 1999
Associate Examiner: Professor D. Sussman	Time: 9:00 A.M. - 1:00 P.M.

INSTRUCTIONS

Calculators are neither required nor permitted.

This exam comprises the cover, two pages of questions and two pages of useful information.