1. (11 marks) Solve the vibrating string problem:

$$\frac{\partial^2\psi}{\partial x^2}=\frac{\partial^2\psi}{\partial t^2};\ 0<x<1,\ t>0$$

(i) $\psi(0,t)=\sin t$, (ii) $\psi(1,t)=0$.

(iii) $\psi(x,0)=0$, (iv) $\left[\frac{\partial\psi}{\partial t}(x,t)\right]_{t=0}=1-x$.

Leave your answer in <u>simplest</u> form.

2. (7 marks) Solve

$$\frac{\partial\psi}{\partial t}=\frac{\partial^2\psi}{\partial x^2}+3;\quad 0<x<\pi,\ t>0$$

(i) $\psi_x(0,t)=0$, (ii) $\psi_x(\pi,t)=0$, (iii) $\psi(x,0)=7$.

3. (12 marks) Solve the following boundary value problem:

$$\nabla^2\psi(r,\theta,z)=0;\ 0\le r<b,\ 0\le\theta<2\pi,\ 0<z<h$$

(i) $\psi(r,\theta,0)=f(r,\theta)$, (ii) $\psi(b,\theta,z)=0$, (iii) $\psi(r,\theta,h)=0$.

4. (a) (6 marks) Solve $\nabla^2\psi(r,\varphi)=0,\ r>a,\ 0\le\varphi\le\pi$.

   (i) $\left[\frac{\partial\psi}{\partial r}\right]_{r=a}=0$, (ii) $\lim_{r\to\infty}[\psi(r,\varphi)-V_0 r\cos\varphi]=0$ and interpret physically.

   (b) (1 mark) Verify the orthogonality of the Legendre polynomials of <u>odd</u> order for the interval $[0,1]$

$$\int_0^1 P_n(x)P_m(x)dx=\frac{\delta nm}{2n+1}.$$

   (c) (6 marks) Find the potential $\psi(r,\varphi)$ in the infinite region $r>b,\ 0<\varphi<\frac{\pi}{2}$, if $\psi=0$ on the plane portion of the boundary $(\varphi=\frac{\pi}{2},\ r>b)$, $\psi\to0$ as $r\to\infty$ and $\psi=f(\cos\varphi)$ on the hemispherical portion of the boundary $(r=b,\ 0\le\varphi<\frac{\pi}{2})$.

   (d) (4 marks) Consider the special case of $f(\cos\varphi)=\cos^3\varphi$.

   You may assume that the general solution of Laplace's equation in spherical coordinates, i.e. $\nabla^2\psi(r,\varphi)=0,\ 0\le\varphi\le\pi,\ \psi$ finite at $\varphi=0$ and $\varphi=\pi$, is given by $\psi(r,\varphi)=\sum^{\infty}\left[A_n r^n+\frac{B_n}{r^{n+1}}\right]P_n(\cos\varphi)$, where P_n are the Legendre polynomials of

6. (10 marks) Solve <u>and</u> interpret physically

$$\nabla^2\psi(r,z) = -q(r,z);\ 0 < r < b,\ 0 < z < \pi,$$

(i) $\psi(r,0) = 0,$ (ii) $\psi(r,\pi) = 0,$ (iii) $\psi(b,z) = 0.$

7. (7 marks) Solve the integral equation

$$\int_{-\infty}^{\infty} \frac{y(\xi)d\xi}{t^2 + 25 - 2t\xi + \xi^2} = \frac{1}{t^2 + 36}.$$

8. (11 marks) Solve

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0, \qquad x > 0,\ y > 0$$

I. $\psi(0,y) = T_0$.

II. $\left.\frac{\partial\psi}{\partial y}(x,y)\right|_{y=0} = h[\psi(x,0) - T_1].$

III. $\lim_{y\to\infty} \psi(x,y) = T_0$ <u>and</u> interpret physically.

9. (13 marks) Solve

$$\begin{aligned} \frac{\partial\psi}{\partial t} &= \frac{\partial^2\psi}{\partial x^2} + h(x,t);\ -\infty < x < \infty,\ t > 0 \\ \psi(x,0) &= f(x). \end{aligned}$$

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-375A

DIFFERENTIAL EQUATIONS

Examiner: Professor C. Roth
Associate Examiner: Professor D. Sussman

Date: Monday, December 7, 1998
Time: 9:00 A.M. - 1:00 P.M.

INSTRUCTIONS

Calculators are neither required nor permitted.

This exam comprises the cover, 2 pages of questions and 1 page of useful information.