MARKS

(3) 1. (a) State and prove the Weak Law of Large Numbers for a sequence of independent and identically distributed random variables $X_1, X_2, \ldots$ with finite mean μ and variance $\sigma^2 < \infty$. Do not prove Chebyshev's inequality.

(2) (b) State the Weak Law of Large Numbers for a sequence of i.i.d. random variables $X_1^2, X_2^2, \ldots$, with finite second moment.

(10) (c) Suppose that a sample of n independent and identically distributed observations, $X_1, X_2, \ldots, X_n$, from some distribution with mean $\mu < \infty$ and unknown variance $\sigma^2 < \infty$, is given. A well-known estimator of σ^2 is given by

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}\right)^2, \quad \text{where} \quad \overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Show that as $n \to \infty$, $s^2 \to \sigma^2$ in probability.
(Hint: $(X_i - \overline{X})^2, i = 1, 2, \ldots$ are NOT i.i.d.)

2. Suppose that $Y_1, Y_2, \ldots$ is a sequence of i.i.d. Poisson random variables with parameter 1.

(5) (a) Derive the moment generating function of Y_i.

(5) (b) Using part (a), show that $S_n = \sum_{i=1}^{n} Y_i$ has a Poisson distribution with parameter n. State any theorems you use.

(2) (c) Show that $P\left(\frac{S_n}{n} \leq 1\right) = \sum_{k=0}^{n} \frac{n^k e^{-n}}{k!}$.

(3) (d) State the Central Limit Theorem for a sequence of i.i.d. r.v.'s with $\mu = \sigma^2 = 1$.

(10) (e) Find $\lim_{n \to \infty} \sum_{k=0}^{n} \frac{n^k e^{-n}}{k!}$.

(5) 3. (a) State and prove Bayes' Theorem.

(10) (b) Suppose that a screening test for AIDS has the following features: (i) If a blood sample actually comes from someone with AIDS then the test will be positive 95% of the time. (ii) If the blood sample comes from someone without AIDS then the test will be negative 95% of the time.
Suppose also that 5% of the population has AIDS. If a blood sample tests positive, what is the probability that the person whose blood was tested has AIDS?

4. Let two random variables, X and Y have joint density

$$f_{X,Y}(x,y) = a + 2bxy, \qquad \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 1$$
$$= 0 \qquad \text{elsewhere}$$

where a and b are constants.

(10) (a) If $E(X) = 1/2$ find a and b. Find $\text{Var}(X)$.

(10) (b) Evaluate $P(Y \le \frac{1}{2}|X = 1/2)$.

(5) (c) Find $E(Y|X = 1/2)$.

(10) 5. Let $\rho(X,Y)$ be the correlation between the two random variables, X and Y with $\mu = 0$, and $\sigma^2 = 1$. Prove that $|\rho(X,Y)| \le 1$ and that $|\rho(X,Y)| = 1$ if and only if $Y = \pm X$ (with probability 1.)

6. The general bivariate Normal density is given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi} \cdot \frac{1}{\sigma_1\sigma_2} \cdot \frac{1}{\sqrt{1-\rho^2}}$$
$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\},$$

for $-\infty < x < \infty,\ -\infty < y < \infty$.

Let X_1 and X_2 be i.i.d. $N(0,1)$ random variables.

(10) (a) Define $Y_1 = X_1$ and $Y_2 = X_1 + X_2$. Show that the joint density of Y_1 and Y_2 is given by

$$f_{Y_1,Y_2}^{(y_1,y_2)} = \frac{1}{2\pi}\exp\left[-\frac{1}{2}(2y_1^2 - 2y_1y_2 + y_2^2)\right].$$

(5) (b) Hence show that Y_1 and Y_2 are bivariate Normal random variables with $\mu_1 = \mu_2 = 0,\ \sigma_1^2 = 1,\ \sigma_2^2 = 2$ and $\rho = 1/\sqrt{2}$.

(5) (c) Are Y_1 and Y_2 independent? Justify your answer in a single short sentence.

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-356A

PROBABILITY

Examiner: Professor D. Wolfson
Associate Examiner: Professor M. Gu

Date: Friday, December 12, 1997
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

Calculators Are Permitted.
Answer all questions.
Although the mark total is 110, you will be marked out of 100. That is, there is a 10 mark bonus.

This exam comprises the cover and 2 pages of questions.