FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS MATH 355

Analysis 4

Examiner: Professor S. W. Drury — Date: Wednesday, April 18, 2007

Associate Examiner: Professor K. N. GowriSankaran — Time: 2: 00 pm. – 5: 00 pm.

INSTRUCTIONS

**Attempt six questions for full credit.**

**This is a closed book examination.**
**Write your answers in the booklets provided.**
**All questions are of equal weight, each is alloted 20 marks.**

**This exam has 7 questions and 10 pages**

1. (i) (4 points) Define the concepts field and $\sigma$-field.
(ii) (2 points) Define the concept of premeasure on a field and measure on a $\sigma$-field.
(iii) (2 points) Define the concept of outer measure.
(iv) (4 points) State the Carathéodory Extension Theorem.
(v) (8 points) If $\mu$ is a premeasure on a field $\mathcal{F}$ of subsets of $X$ and $\mu^\star$ is the outer measure it defines on $X$ by the equation $\mu^\star(A) = \inf \sum_{j=1}^{\infty} \mu(A_j)$ where the infimum is taken over all possible sequences of sets $A_j \in \mathcal{F}$ such that $A \subseteq \bigcup_{j=1}^{\infty} A_j$, show that for any subsets $A$ and $B$ of $X$ that $\mu^\star(A \cup B) + \mu^\star(A \cap B) \le \mu^\star(A) + \mu^\star(B)$.

**Solution:**
(i) Let $X$ be a set. Then a collection $\mathcal{F}$ of subsets of $X$ is a field if and only if

(a) $X \in \mathcal{F}$.

(b) $A \in \mathcal{F} \Longrightarrow X \setminus A \in \mathcal{F}$.

(c) $A \in \mathcal{F}, B \in \mathcal{F} \Longrightarrow A \cup B \in \mathcal{F}$.

Let $X$ be a set. Then a collection $\mathcal{F}$ of subsets of $X$ is a $\sigma$-field if and only if

(a) $X \in \mathcal{F}$.

(b) $A \in \mathcal{F} \Longrightarrow X \setminus A \in \mathcal{F}$.

(c) $A_k \in \mathcal{F}$ for $k \in K$, $K$ countable $\Longrightarrow \bigcup_{k \in K} A_k \in \mathcal{F}$.

(ii) We now define the concept of a measure (premeasure) on a $\sigma$-field (field) $\mathcal{F}$ of subsets of $X$ as a function $\mu : \mathcal{F} \longrightarrow [0, \infty]$ such that

(a) $\mu(\emptyset) = 0$.

(b) $\mu\left(\bigcup_{k \in K} A_k\right) = \sum_{k \in K} \mu(A_k)$ whenever $K$ is a countable index set and $A_k$ are *pairwise disjoint* subsets of $X$ with $A_k \in \mathcal{F}$ and $\bigcup_{k \in K} A_k \in \mathcal{F}$.

**Note:** Tragically many students defined a premeasure as a finitely additive set function. This is incorrect.

(iii) An outer measure $\theta$ on a set $X$ is a map $\theta : \mathcal{P}_X \longrightarrow [0, \infty]$ with the following properties

(a) $\theta(\emptyset) = 0$.

(b) If $A \subseteq B \subseteq X$, then $\theta(A) \le \theta(B)$.

(c) $\theta\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_{j=1}^{\infty} \theta(A_j)$.

(iv) Let $\mu$ be a premeasure on a field $\mathcal{F}$ of subsets of $X$. Let $\mathcal{G}$ be the $\sigma$-field generated by $\mathcal{F}$. Then there exists a measure $\nu$ on $\mathcal{G}$ which agrees with $\mu$ on $\mathcal{F}$.

(v) The easiest way is to use the Carathéodory Extension Theorem. Let the extension be $(X, \mathcal{G}, \nu)$. Then clearly we have $\mu^*(A) = \inf_{A \subseteq G \in \mathcal{G}} \nu(G)$, an equivalent way of rewriting the definition of $\mu^*$ a posteriori. Now given $\epsilon > 0$ we can find $P, Q \in \mathcal{G}$ such that $A \subseteq P$, $B \subseteq Q$, $\nu(P) < \mu^*(A)+\epsilon$ and $\nu(Q) < \mu^*(B) + \epsilon$

From the three identities

$$\begin{aligned} \nu(P \cup Q) &= \nu(P \setminus Q) + \nu(P \cap Q) + \nu(Q \setminus P), \\ \nu(P) &= \nu(P \setminus Q) + \nu(P \cap Q), \\ \nu(Q) &= \nu(P \cap Q) + \nu(Q \setminus P). \end{aligned}$$

we get

$$\nu(P) + \nu(Q) = \nu(P \setminus Q) + \nu(P \cap Q) + \nu(P \cap Q) + \nu(Q \setminus P) = \nu(P \cup Q) + \nu(P \cap Q).$$

But $A \cup B \subseteq P \cup Q$ and $A \cap B \subseteq P \cap Q$ so that

$$\mu^\star(A \cup B) + \mu^\star(A \cap B) \le \nu(P \cup Q) + \nu(P \cap Q) = \nu(P) + \nu(Q) < \mu^*(A) + \mu^*(B) + 2\epsilon.$$

Letting $\epsilon \downarrow 0$ gives the desired result.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) (5 points) Under what conditions can one define $\int f(x) d\mu(x)$ for a signed $\mathcal{M}$-measurable function $f$ on $X$? In this case give the definition in terms of the integral of nonnegative $\mathcal{M}$-measurable functions on $X$.

Let $g$ be a nonnegative $\mathcal{M}$-measurable function on $X$ satisfying $\int g(x) d\mu(x) < \infty$.

(ii) (5 points) Prove Tchebychev's inequality $\mu(\{x; g(x) > t\}) \le \dfrac{1}{t} \displaystyle\int g(x) d\mu(x)$ for $t > 0$.

(iii) (10 points) Let $\mu(X) = 1$ and let $f$ be a signed $\mathcal{M}$-measurable function such that $\int f d\mu = 0$ and $\int f^2 d\mu = 1$. Show that $\mu(\{x; f(x) > s\}) \le \dfrac{1}{1+s^2}$ for $s > 0$.

*Hint:* Consider $g(x) = (sf(x) + 1)^2$.

**Solution:**

(i) The integral is only defined in case $\int |f| d\mu < \infty$. In this case, we define

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) \ge 0, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{and} \qquad f_-(x) = \begin{cases} -f(x) & \text{if } f(x) \le 0, \\ 0 & \text{otherwise.} \end{cases}$$

In this way, we see that $f_+ \geq 0$, $f_- \geq 0$ and $f = f_+ - f_-$. Now, it is clear that $f_\pm \leq |f|$ and that $\int f_\pm d\mu < \infty$. The definition $f d\mu = \int f_+ d\mu - \int f_- d\mu$ makes sense as the difference of two finite nonnegative numbers.

(ii) Let $A = \{x; g(x) > t\}$, then $A \in \mathcal{M}$. Since $g$ is nonegative and $g \geq t\mathbb{1}_A$, it follows that

$$\int g d\mu \geq \int t\mathbb{1}_A d\mu = t\mu(A)$$

as required.

(iii) Using the hypotheses we have

$$\int g d\mu = \int (sf+1)^2 d\mu = s^2 \int f^2 d\mu + 2s \int f d\mu + \mu(X) = s^2 + 1$$

But $g$ is nonnegative and $f(x) > s \Longrightarrow g(x) > (s^2+1)^2$, so taking $t = (s^2+1)^2$, we get

$$\mu(\{x; f(x) > s\}) \leq \frac{s^2+1}{(s^2+1)^2} = \frac{1}{s^2+1}$$

3. (i) (5 points) State the Monotone Convergence Theorem.

(ii) (5 points) State the Dominated Convergence Theorem.

(iii) (10 points) Find $\lim_{n\to\infty} n \int_0^\infty \frac{1}{1+x^4} \sin\left(\frac{x}{n}\right) dx$. In answering the question you may use the inequality $|\sin(u)| \leq \min(1, |u|)$. Otherwise, justify all steps and for full credit simplify your answer as much as possible.

**Solution:**

(i) If $f_n, f$ are nonnegative measurable functions and if $f_n \uparrow f$ pointwise, then $\int f_n d\mu \uparrow \int f d\mu$.

(ii) Let $f_n$ be a sequence of measurable functions and suppose that $f_n \longrightarrow f$ pointwise. Further suppose that there is a (nonnegative) function $g$ such that $|f_n| \leq g$ pointwise for every $n \in \mathbb{N}$. If $\int g d\mu < \infty$, then necessarily

$$\int f_n d\mu \underset{n\to\infty}{\longrightarrow} \int f d\mu.$$

(iii) From the given inequality, $n\left|\sin\left(\frac{x}{n}\right)\right| \leq x$ for $x \geq 0$ and we know from L'Hôpital's Rule that $n \sin\left(\frac{x}{n}\right) \longrightarrow x$ as $n \to \infty$. Letting $f_n(x) = n\frac{1}{1+x^4}\sin\left(\frac{x}{n}\right)$ we can take $f(x) = g(x) = \frac{x}{1+x^4}$ in the Dominated Convergence Theorem. The value of the limit is

$$\int_0^\infty \frac{x}{1+x^4} dx = \int_0^\infty \frac{\frac{1}{2}u}{1+u^2} du = \frac{\pi}{4}.$$

4. (i) (5 points) Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces. Define $\mathcal{S} \otimes \mathcal{T}$.
If $X$ is a metric space, we denote $\mathcal{B}_X$, its Borel $\sigma$-field.
(ii) (15 points) Prove in detail that $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$.

**Solution:**
(i) A measurable rectangle is a subset of $X \times Y$ of the form $S \times T$ with $S \in \mathcal{S}$ and $T \in \mathcal{T}$. We define $\mathcal{S} \otimes \mathcal{T}$ to be the smallest $\sigma$-field containing the measurable rectangles.
(ii) To see that $\mathcal{B}_{\mathbb{R}^2} \subseteq \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$, recall that every open subset of $\mathbb{R}^2$ is a countable union of open rectangles $J \times K$ where $J, K$ are open intervals in $\mathbb{R}$. This shows that every open subset of $\mathbb{R}^2$ lies in the $\sigma$-field $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$. The inclusion now follows from the definition of $\mathcal{B}_{\mathbb{R}^2}$. The other direction is easier, but more involved. One starts from

$$A, B \text{ open} \implies A \times B \text{ open} \implies A \times B \in \mathcal{B}_{\mathbb{R}^2}.$$

Now let $A$ be a fixed open set then clearly

$$\{B; B \subseteq \mathbb{R}, A \times B \in \mathcal{B}_{\mathbb{R}^2}\} \text{ is a } \sigma\text{-field on } \mathbb{R} \text{ containing the open sets.}$$

It follows that

$$A \text{ open }, B \text{ borel} \implies A \times B \in \mathcal{B}_{\mathbb{R}^2}.$$

Then, fix $B$ borel the clearly

$$\{A; A \subseteq \mathbb{R}, A \times B \in \mathcal{B}_{\mathbb{R}^2}\} \text{ is a } \sigma\text{-field on } \mathbb{R} \text{ containing the open sets.}$$

We may deduce that

$$A, B \text{ borel} \implies A \times B \in \mathcal{B}_{\mathbb{R}^2}.$$

Finally, since $\mathcal{B}_{\mathbb{R}^2}$ is a $\sigma$-field, $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} \subseteq \mathcal{B}_{\mathbb{R}^2}$.

5. (i) (5 points) State Tonelli's Theorem.
(ii) (5 points) State Fubini's Theorem.
(iii) (10 points) Starting from the identity

$$\int_0^\infty e^{-sx}\sin(ux)dx = \frac{u}{u^2+s^2}$$

valid for $s > 0$ and $u \in \mathbb{R}$, show that

$$\int_0^\infty e^{-sx}\frac{1-\cos(tx)}{x}dx = \tfrac{1}{2}\ln(s^2+t^2) - \ln(s)$$

provided that $s > 0$ and $t \in \mathbb{R}$. *Hint:* $\displaystyle\int_0^t \sin(ux)du = \frac{1-\cos(tx)}{x}$.

**Solution:**

(i) Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be $\sigma$-finite measure spaces. Let $f : X \times Y \longrightarrow [0, \infty]$ be $\mathcal{S} \otimes \mathcal{T}$ measurable. Then $\varphi(x) = \int f(x, y)d\nu(y)$ and $\psi(y) = \int f(x, y)d\mu(x)$ define nonnegative measurable functions on $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ respectively and

$$\int \varphi(x)d\mu(x) = \int f(x, y)d(\mu \times \nu)(x, y) = \int \psi(y)d\nu(y).$$

(ii) Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be $\sigma$-finite measure spaces. Let $f : X \times Y \longrightarrow \mathbb{R}$ be $\mathcal{S} \otimes \mathcal{T}$ measurable. Suppose that one of the three quantities

$$\iint |f(x, y)|d\nu(y)d\mu(x) = \int |f|d(\mu \times \nu) = \iint |f(x, y)|d\mu(x)d\nu(y).$$

is finite (they are all equal by Tonelli's Theorem). Then

$$\varphi(x) = \int f(x, y)d\nu(y) \text{ and } \psi(y) = \int f(x, y)d\mu(x)$$

almost everywhere (w.r.t $\mu$ and $\nu$ respectively) define measurable functions on $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ respectively, the integrals being absolutely convergent at almost every point, and

$$\int \varphi(x)d\mu(x) = \int f(x, y)d(\mu \times \nu)(x, y) = \int \psi(y)d\nu(y),$$

where these integrals also make sense as absolutely convergent integrals.

(iii) Since $t \mapsto \frac{1}{2}\ln(s^2 + t^2) - \ln(s)$ and $t \mapsto 1 - \cos(tx)$ are even functions, there is no loss in assuming that $t \geq 0$.

Assuming that Fubini's Theorem can be applied, we have

$$\begin{aligned}\int_0^\infty e^{-sx}\frac{1-\cos(tx)}{x}dx &= \int_0^\infty \int_0^t e^{-sx}\sin(ux)dudx \\ &= \int_0^t \int_0^\infty e^{-sx}\sin(ux)dxdu \\ &= \int_0^t \frac{u}{u^2+s^2}du \\ &= \tfrac{1}{2}\ln(s^2+t^2)-\ln(s)\end{aligned}$$

To justify, the measure spaces are certainly $\sigma$-finite, the integrand $(x,u)\mapsto e^{-sx}\sin(ux)$ is continuous and hence Borel and we have

$$\int_0^t\int_0^\infty |e^{-sx}\sin(ux)|dxdu \le \int_0^t\int_0^\infty e^{-sx}dxdu = \frac{t}{s} < \infty$$

for $s > 0$.

6. Let $\mathcal{L}$ be the Lebesgue $\sigma$-field on $[0,\infty[$ and $d\mu(x) = e^{-x}dx$. Consider the linear subspace $M$ of $H = L^2([0,\infty[,\mathcal{L},\mu)$ consisting of equivalence classes of functions that are periodic a.e. with period $2\pi$, i.e.

$$f(x+2\pi) = f(x) \text{ a.a. } x\in[0,\infty[$$

(i) (4 points) Show that $M$ is itself an $L^2$ space over a smaller $\sigma$-field than $\mathcal{L}$.
(ii) (4 points) Deduce that $M$ is a *closed* linear subspace of $H$. What fact are you using here?
(iii) (4 points) Show that for $f,g\in H$,

$$\langle f,g\rangle = \sum_{k=0}^\infty e^{-2k\pi}\int_0^{2\pi}\overline{f(x+2k\pi)}g(x+2k\pi)e^{-x}dx$$

(iv) (4 points) Show that the closed linear span of the functions $x\mapsto e^{inx}$ as $n$ runs over all integers is the whole of $M$. What fact are you using here?
(v) (4 points) For an arbitrary member $f$ of $H$, let $h$ be its orthogonal projection on $M$. Show that

$$h(x) = (1-e^{-2\pi})\sum_{k=0}^\infty e^{-2k\pi}f(x+2k\pi),$$

for almost all $x$ in $[0,2\pi[$ (and extended by periodicity for other values of $x$).

**Solution:**

(i) Let $\mathcal{G}$ be the $\sigma$-field of Lebesgue measurable subsets of $[0,\infty[$ that are a.e. periodic with period $2\pi$. Then it is routine to check that $\mathcal{G}$ is a $\sigma$-field and that $M = L^2([0,\infty[,\mathcal{G},\mu)$.

(ii) Since $M$ is an $L^2$ space, it is complete and hence is closed in any metric space that contains it isometrically, such as $L^2([0,\infty[,\mathcal{L},\mu)$.

(iii)

$$\begin{aligned}\langle f,g\rangle &= \int_0^\infty \overline{f(x)}g(x)e^{-x}dx \\ &= \sum_{k=0}^\infty \int_{2k\pi}^{2(k+1)\pi} \overline{f(x)}g(x)e^{-x}dx\end{aligned}$$

by splitting up the range of integration and using Dominated Convergence,

$$= \sum_{k=0}^\infty \int_0^{2\pi} \overline{f(x+2k\pi)}g(x+2k\pi)e^{-x-2k\pi}dx$$

by changing variables in each of the inner integrals

$$= \sum_{k=0}^\infty e^{-2k\pi}\int_0^{2\pi} \overline{f(x+2k\pi)}g(x+2k\pi)e^{-x}dx$$

(iv) The trigonometric system $e_n(x) = e^{inx}$ (for $n \in \mathbb{Z}$) is an orthonormal basis for $L^2([0,2\pi[,\mathcal{L},dx/2\pi)$ and so its closed linear span is the whole of $L^2([0,2\pi[,\mathcal{L},dx/2\pi)$. However, continuing from (iii) above, we have for $f \in M$

$$\|f\|_H^2 = \frac{1}{1-e^{-2\pi}}\int_0^{2\pi} |f(x)|^2 e^{-x}dx$$

So

$$C_1\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2dx \le \|f\|_H^2 \le C_2\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2dx$$

for suitable $C_1$, $C_2$ satisfying $0 < C_1 < C_2 < \infty$. Thus the restriction mapping $f \mapsto f|_{[0,2\pi[}$ is a one-to-one linear map of $M$ onto $L^2([0,2\pi[,\mathcal{L},dx/2\pi)$. Furthermore on $M$ the metric coming from $H$ and the metric coming from $L^2([0,2\pi[,\mathcal{L},dx/2\pi)$ are equivalent and have the same convergent sequences and therefore the same closed sets. It follows that $M$ is also the closed linear span of $e_n(x) = e^{inx}$ (for $n \in \mathbb{Z}$) in the metric coming from $H$.

**Note:** The set of functions $x \mapsto e^{inx}$ as $n$ runs over all integers is not an orthonormal basis of $M$.

(v) Let $g \in M$. Then $g \perp f - h$ or

$$0 = \sum_{k=0}^\infty e^{-2k\pi}\int_0^{2\pi} \overline{g(x)}\Big(f(x+2k\pi) - h(x)\Big)e^{-x}dx$$

using the periodicity of $g$ and $h$. Applying Fubini's Theorem, we get

$$0 = \int_0^{2\pi} \overline{g(x)} \left( \left( \sum_{k=0}^{\infty} e^{-2k\pi} f(x + 2k\pi) \right) - \frac{1}{1 - e^{-2\pi}} h(x) \right) e^{-x} dx.$$

Fubini's Theorem is valid since $g$, $h$ and $x \mapsto \sum_{k=0}^{\infty} e^{-2k\pi} |f(x + 2k\pi)|$ are all $L^2$ functions on $[0, 2\pi[$ and the integrand is therefore dominated by a nonnegative $L^1$ function by the Cauchy–Schwarz Inequality. (Dominated convergence can also be used here). Finally, since $g$ is arbitrary, we find

$$h(x) = (1 - e^{-2\pi}) \sum_{k=0}^{\infty} e^{-2k\pi} f(x + 2k\pi),$$

for almost all $x$ in $[0, 2\pi[$.

7. Consider the trigonometric polynomials $P_m$ and $Q_m$ defined for nonnegative integers $m$ inductively as follows

$$P_0(t) = Q_0(t) = 1 \text{ and } P_{m+1}(t) = P_m(t) + e^{i2^m t} Q_m(t), \; Q_{m+1}(t) = P_m(t) - e^{i2^m t} Q_m(t)$$

(i) (5 points) Show that $P_1(t) = 1 + e^{it}$, $Q_1(t) = 1 - e^{it}$, $P_2(t) = 1 + e^{it} + e^{2it} - e^{3it}$ and $Q_2(t) = 1 + e^{it} - e^{2it} + e^{3it}$.

(ii) (5 points) Show that $\widehat{P_m}(n) = 0$ if $n < 0$ or if $n \geq 2^m$ and that $\widehat{P_m}(n) = 1$ or $-1$ otherwise.

(iii) (5 points) Show that $|P_{m+1}(t)|^2 + |Q_{m+1}(t)|^2 = 2\left(|P_m(t)|^2 + |Q_m(t)|^2\right)$ and deduce first that $|P_m(t)|^2 + |Q_m(t)|^2 = 2^{m+1}$ for all $t$ and then that $\sup_t |P_m(t)| \leq 2^{\frac{m+1}{2}}$.

(iv) (5 points) Show that $\int_0^{2\pi} |P_m(t)|^2 dt = 2^{m+1}\pi$.

**Note:** This question had a small error which has been corrected in this version.

**Solution:**

(i) According to the definitions, $P_1 = P_0 + e^{it} Q_0 = 1 + e^{it}$, $Q_1 = P_0 - e^{it} Q_0 = 1 - e^{it}$, $P_2 = P_1 + e^{2it} Q_1 = 1 + e^{it} + e^{2it} - e^{3it}$ and $Q_2 = P_1 - e^{2it} Q_1 = 1 + e^{it} - e^{2it} + e^{3it}$.

(ii) Proof by induction using the induction hypothesis that

- $\widehat{P_m}(n) = 0$ if $n < 0$ or if $n \geq 2^m$ and that $\widehat{P_m}(n) = 1$ or $-1$ otherwise.
- $\widehat{Q_m}(n) = 0$ if $n < 0$ or if $n \geq 2^m$ and that $\widehat{Q_m}(n) = 1$ or $-1$ otherwise.

which clearly starts correctly. We have

$$\widehat{P_m}(n) = \widehat{P_{m-1}}(n) + \widehat{Q_{m-1}}(n - 2^{m-1})$$
$$\widehat{Q_m}(n) = \widehat{P_{m-1}}(n) - \widehat{Q_{m-1}}(n - 2^{m-1})$$

and we check using the induction hyothesis that $P$ (respectively $Q$) satisfies

$$\widehat{P_m}(n) = \begin{cases} 0 & \text{if } n < 0 \text{ since } \widehat{P_{m-1}}(n) = \widehat{Q_{m-1}}(n - 2^{m-1}) = 0, \\ \pm 1 & \text{if } 0 \le n < 2^{m-1} \text{ since } \widehat{P_{m-1}}(n) = \pm 1,\ \widehat{Q_{m-1}}(n - 2^{m-1}) = 0, \\ \pm 1 & \text{if } 2^{m-1} \le n < 2^m \text{ since } \widehat{P_{m-1}}(n) = 0,\ \widehat{Q_{m-1}}(n - 2^{m-1}) = \pm 1, \\ 0 & \text{if } n \ge 2^m \text{ since } \widehat{P_{m-1}}(n) = \widehat{Q_{m-1}}(n - 2^{m-1}) = 0, \end{cases}$$

(iii) We have

$$\begin{aligned} &|P_{m+1}(t)|^2 + |Q_{m+1}(t)|^2 \\ &= \left|P_m(t) + e^{i2^m t} Q_m(t)\right|^2 + \left|P_m(t) - e^{i2^m t} Q_m(t)\right|^2 \\ &= |P_m(t)|^2 + |Q_m(t)|^2 + 2\Re\Big(\overline{P_m(t)} e^{i2^m t} Q_m(t)\Big) + |P_m(t)|^2 + |Q_m(t)|^2 - 2\Re\Big(\overline{P_m(t)} e^{i2^m t} Q_m(t)\Big) \\ &= 2\Big(|P_m(t)|^2 + |Q_m(t)|^2\Big) \end{aligned}$$

and a simple induction argument gives the required conclusion. Since $|P_m(t)|^2 + |Q_m(t)|^2 = 2^{m+1}$, it follows that $|P_m(t)|^2 \le 2^{m+1}$ for all $t$ and hence $|P_m(t)| \le 2^{\frac{m+1}{2}}$.

(iv) From the Plancherel Theorem, $\displaystyle\int_0^{2\pi} |P_m(t)|^2 dt = 2\pi \sum_{n \in \mathbb{Z}} \left|\widehat{P_m}(n)\right|^2 = 2^{m+1}\pi$, from (ii) above.

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