FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS MATH 355

Analysis 4

Examiner: Professor S. W. Drury   Date: Tuesday, April 18, 2006

Associate Examiner: Professor K. N. GowriSankaran   Time: 2: 00 pm. – 5: 00 pm.

INSTRUCTIONS

Attempt six questions for full credit.

This is a closed book examination.
Write your answers in the booklets provided.
All questions are of equal weight, each is alloted 20 marks.

This exam has 7 questions and 9 pages

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1. (i) (5 points) Define the concept σ-*field*.
    (ii) (5 points) Define the concept *measure* on a σ-field.
    (iii) (5 points) Let Y be a subset of X. For any family $\mathcal{M}$ of subsets of X, denote by $\mathcal{M}_Y$ the family $\mathcal{M}_Y = \{M \cap Y; M \in \mathcal{M}\}$ of subsets of Y. If $\mathcal{M}$ is a σ-field of subsets of X, show that $\mathcal{M}_Y$ is σ-field on Y. Note that you are not allowed to assume that $Y \in \mathcal{M}$.
    (iv) (5 points) If μ is a measure on $(Y, \mathcal{M}_Y)$, does $\nu(M) = \mu(M \cap Y)$ define a measure on $(X, \mathcal{M})$? Justify your answer.

Solution:
    (i) Let X be a set. Then a collection $\mathcal{F}$ of subsets of X is a σ-field if and only if

(a) $X \in \mathcal{F}$.

(b) $A \in \mathcal{F} \implies X \setminus A \in \mathcal{F}$.

(c) $A_k \in \mathcal{F}$ for $k \in K$, K countable $\implies \bigcup_{k \in K} A_k \in \mathcal{F}$.

   (ii) A measure on a σ-field $\mathcal{F}$ of subsets of X as a function $\mu : \mathcal{F} \longrightarrow [0, \infty]$ such that

(a) $\mu(\emptyset) = 0$.

(b) $\mu\left(\bigcup_{k \in K} A_k\right) = \sum_{k \in K} \mu(A_k)$ whenever K is a countable index set and A_k are pairwise disjoint subsets of X with $A_k \in \mathcal{F}$ and $\bigcup_{k \in K} A_k \in \mathcal{F}$.

   (iii)

(a) Since $X \in \mathcal{M}$, $Y = X \cap Y \in \mathcal{M}_Y$.

(b) Let $B \in \mathcal{M}_Y$. Then there exists $A \in \mathcal{M}$ such that $B = A \cap Y$. But $Y \setminus B = (X \setminus A) \cap Y \in \mathcal{M}_Y$, since $(X \setminus A) \in \mathcal{M}$.

(c) Let $B_k \in \mathcal{M}_Y$ for $k \in \mathbb{N}$, then there exist $A_k \in \mathcal{M}$ with $B_k = A_k \cap Y$. But now we find $\bigcup_{k=1}^{\infty} B_k = \left(\bigcup_{k=1}^{\infty} A_k\right) \cap Y$ and it follows that $\bigcup_{k=1}^{\infty} B_k \in \mathcal{M}_Y$ since $\bigcup_{k=1}^{\infty} A_k \in \mathcal{M}$.

   (iv)

(a) $\nu(\emptyset) = \mu(\emptyset) = 0$.

(b) Let A_k be pairwise disjoint subsets of X in $\mathcal{M}$, then $B_k = A_k \cap Y$ are pairwise disjoint subsets of Y in $\mathcal{M}_Y$. Further, as in (c) above $\bigcup_{k=1}^{\infty} B_k = \left(\bigcup_{k=1}^{\infty} A_k\right) \cap Y$ and consequently

$$\nu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\left(\bigcup_{k=1}^{\infty} A_k\right) \cap Y\right) = \mu\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(B_k) = \sum_{k=1}^{\infty} \nu(A_k).$$

2. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (i) (5 points) Under what conditions can one define $\int f(x)d\mu(x)$ for a signed $\mathcal{M}$-measurable function f on X? In this case give the definition in terms of the integral of nonnegative $\mathcal{M}$-measurable functions on X.

   Let g be a nonnegative $\mathcal{M}$-measurable function on X satisfying $\int g(x)d\mu(x) < \infty$.

   (ii) (5 points) Prove Tchebychev's inequality $\mu(\{x; g(x) > t\}) \leq \frac{1}{t}\int g(x)d\mu(x)$ for $t > 0$.

   (iii) (5 points) If $f \in L^p(X, \mathcal{M}, \mu)$ where $1 \leq p < \infty$, show that the inequality

$$\mu(\{x; |f(x)| > t\}) \leq t^{-p}\|f\|_p^p$$

holds for every $t > 0$.

   (iv) (5 points) For every p with $1 \leq p < \infty$, find a measure space $(X, \mathcal{M}, \mu)$ and a measurable function f on $(X, \mathcal{M})$ such that $\mu(\{x; |f(x)| > t\}) \leq t^{-p}$ holds for every $t > 0$ but $f \notin L^p(X, \mathcal{M}, \mu)$.

Solution:

   (i) The integral is only defined if $\int |f(x)|d\mu(x) < \infty$ and in this case we set

$$\int f(x)d\mu(x) = \int f_+(x)d\mu(x) - \int f_-(x)d\mu(x)$$

where $f_\pm(x) = \max(0, \pm f(x))$.

   (ii) Let $E_t = \{x; g(x) > t\}$, then we have

$$\int g(x)d\mu(x) \geq \int t \mathbb{1}_{E_t} d\mu(x) = t\mu(E_t),$$

so that $\mu(E_t) \leq \frac{1}{t}\int g(x)d\mu(x)$ as required.

   (iii) Applying the previous part of the question to $g = |f|^p$ we have, since $\{x; g(x) > t^p\} = \{x; |f(x)| > t\}$ that

$$\mu(\{x; |f(x)| > t\}) \leq \mu(\{x; |g(x)| > t^p\}) \leq t^{-p}\int g(x)d\mu(x) = \|f\|_p^p t^{-p}.$$

   (iv) Put $f(x) = x^{-\frac{1}{p}}$ on $]0, \infty[$ with Lebesgue measure and then $\{x; |f(x)| > t\} =]0, t^{-p}[$ has measure exactly t^{-p}, but $\int_0^\infty |f(x)|^p dx = \int_0^\infty x^{-1}dx = \infty$, so $f \notin L^p(]0, \infty[)$.

3. (20 points)   State whether the following statement is true or false and give either a proof or an explicit counterexample.

Every closed subset of the real line with empty interior has zero Lebesgue measure.

Solution:
    This is false. The easiest solution is as follows. Let $(q_k)_{k=1}^{\infty}$ be an enumeration of the points of $\mathbb{Q} \cap [0, 1]$. Let $X = \bigcup_{k=1}^{\infty}]q_k - 2^{-2-k}, q_k + 2^{-2-k}[$. Then X is a union of open intervals and hence is open. The measure of X is bounded by $\sum_{k=1}^{\infty} 2^{-1-k} = \frac{1}{2}$. Hence $[0, 1] \setminus X$ is a closed set of measure at least $\frac{1}{2}$. But if $[0, 1] \setminus X$ contains an interval of strictly positive length, there is a rational number $q \in [0, 1] \setminus X$. But this is impossible since then $q = q_k \in X$ for some $k \in \mathbb{N}$.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) (10 points) Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued $\mathcal{M}$-measurable functions on X. Let $A = \{x \in X; \lim_{n \to \infty} f_n(x) \text{ exists}\}$. Show from first principles that $A \in \mathcal{M}$.

(ii) (10 points) If $A_n \in \mathcal{M}$ for $n = 1, 2, 3, \ldots$ and if $\displaystyle\sum_{n=1}^{\infty} \mu(A_n) < \infty$, show that μ-almost every point of X lies in only finitely many of the A_n.

Solution:

(i) We show first that if $(f_n)_{n=1}^{\infty}$ is a sequence of $\mathcal{M}$-measurable functions on X taking values in the extended real line $[-\infty, \infty]$, then $\sup_n f_n$ is also $\mathcal{M}$-measurable. This is a consequence of the identity

$$\{x; \sup_{n=1}^{\infty} f_n(x) > t\} = \bigcup_{n=1}^{\infty} \{x; f_n(x) > t\}$$

and the fact that the intervals $]t, \infty]$ generate all Borel subsets of $[-\infty, \infty]$. An exactly similar argument shows that $\inf_n f_n$ is $\mathcal{M}$-measurable. It now follows that

$$\limsup_{n \to \infty} f_n(x) = \inf_{m=1}^{\infty} \sup_{n=m}^{\infty} f_n(x) \qquad \text{and} \qquad \liminf_{n \to \infty} f_n(x) = \sup_{m=1}^{\infty} \inf_{n=m}^{\infty} f_n(x)$$

are also $\mathcal{M}$-measurable. Now, interpreting the phrase "$\lim_{n \to \infty} f_n(x)$ exists" to mean that the limit exists in $\mathbb{R}$ (as opposed to $[-\infty, \infty]$), we see that

$$X \setminus A = \{x; \limsup_{n \to \infty} f_n(x) = -\infty\} \cup \{x; \liminf_{n \to \infty} f_n(x) = \infty\} \cup \{x; \liminf_{n \to \infty} f_n(x) < \limsup_{n \to \infty} f_n(x)\}.$$

The first two of these sets are seen to be measurable and the third can be written

$$\{x; \liminf_{n \to \infty} f_n(x) < \limsup_{n \to \infty} f_n(x)\} = \bigcup_{q \in \mathbb{Q}} \left(\{x; \liminf_{n \to \infty} f_n(x) < q\} \cap \{x; q < \limsup_{n \to \infty} f_n(x)\} \right)$$

and hence is also measurable. It follows that $X \setminus A$ and hence A is measurable.

(ii) The set E of x that belong to infinitely many of the A_n can be written as $\displaystyle\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n$. Thus, for all $m \in \mathbb{N}$,

$$\mu(E) \leq \mu\left(\bigcup_{n=m}^{\infty} A_n \right) \leq \sum_{n=m}^{\infty} \mu(A_n) \xrightarrow[m \to \infty]{} 0,$$

since $\displaystyle\sum_{n=1}^{\infty} \mu(A_n) < \infty$. It follows that $\mu(E) = 0$. So, for almost all x, x lies in only finitely many of the A_n.

5. Let μ be Lebesgue measure on $[0, 1]$.

   (i) (6 points) Show that the functions $f_k(t) = \sqrt{2}\cos(k\pi t)$ form an orthonormal set in the Hilbert space $H = L^2([0, 1], \mathcal{B}_{[0,1]}, \mu)$ as k runs over the positive integers.

   (ii) (4 points) Let M be the closed linear span of $\{f_k; k = 1, 2, 3 \ldots\}$. Find $g \in H$, such that $g \perp f_k$ for all $k \in \mathbb{N}$ and deduce that M is not the whole of H.

   (iii) (6 points) Let $f(t) = \sqrt{2}\sin(\pi t)$ and P the orthogonal projection onto M. Find a series expansion (convergent in H) for $P(f)$ in the form

$$P(f) = \sum_{k=1}^{\infty} a_k f_k$$

computing the a_k explicitly.

   (iv) (4 points) By comparing the norms of f and $P(f)$, show that $f \notin M$.

Solution:

   (i) We have

$$\|f_k\|_2^2 = 2 \int_0^1 (\cos(k\pi t))^2 dt = \int_0^1 \Big(1 + \cos(2k\pi t)\Big) dt = \left[t + \frac{\sin(2k\pi t)}{2k\pi}\right]_0^1 = 1$$

and for $k \neq \ell$

$$\langle f_k, f_\ell \rangle = 2 \int_0^1 \cos(k\pi t)\cos(\ell\pi t) dt = \int_0^1 \Big(\cos((k - \ell)\pi t) + \cos((k + \ell)\pi t)\Big) dt$$

$$= \left[\frac{\sin((k - \ell)\pi t)}{(k - \ell)\pi} + \frac{\sin((k + \ell)\pi t)}{(k + \ell)\pi}\right]_0^1 = 0$$

showing that $(f_k)_{k=1}^{\infty}$ is an orthonormal set.

   (ii) Let $g = \mathbb{1}_{[0,1]} \in H$ and then it is clear that

$$\langle f_k, g \rangle = \sqrt{2} \int_0^1 \cos(k\pi t) dt = \sqrt{2} \left[\frac{\sin(k\pi t)}{k\pi}\right]_0^1 = 0$$

so that $g \perp f_k$ for all $k \in \mathbb{N}$. Extending by linearity and continuity, we see that $g \perp M$, that $g \notin M$ and that $M \subset H$.

(iii) We obtain

$$a_k = \langle f, f_k \rangle = 2 \int_0^1 \cos(k\pi t)\sin(\pi t)dt = \int_0^1 \Big(\sin((k+1)\pi t) - \sin((k-1)\pi t)\Big)dt$$

$$= \left[\frac{\cos((k-1)\pi t)}{(k-1)\pi} - \frac{\cos((k+1)\pi t)}{(k+1)\pi} \right]_0^1 = -\frac{2(1+(-1)^k)}{(k^2-1)\pi}$$

$$= \begin{cases} -\dfrac{4}{(k^2-1)\pi} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

Now we also have

$$\|f\|_2^2 = 2 \int_0^1 (\sin(\pi t))^2 dt = \int_0^1 \Big(1 - \cos(2\pi t)\Big)dt = \left[t - \frac{\sin(2\pi t)}{2\pi} \right]_0^1 = 1$$

So, that if $f \in M$ then $f = P(f)$ and $\|f\| = \|P(f)\|$. We can compute (using Maple)

$$\|P(f)\|^2 = \sum_{k=1}^{\infty} |a_k|^2 = \sum_{\ell=1}^{\infty} \frac{16}{\pi^2(4\ell^2-1)^2} = \frac{\pi^2-8}{\pi^2} < 1 = \|f\|^2$$

and it follows that $f \notin M$. The exact summation is difficult to arrive at and is easily circumvented with estimates. We have $4\ell^2 - 1 \ge 3\ell^2$ for $\ell \ge 1$, leading to

$$\sum_{\ell=1}^{\infty} \frac{16}{\pi^2(4\ell^2-1)^2} \le \frac{16}{9\pi^2} \sum_{\ell=1}^{\infty} \ell^{-4} \le \frac{16}{9\pi^2}\left(1 + \int_1^{\infty} x^{-4}dx\right) = \frac{64}{27\pi^2} < 1.$$

using the integral test to make the estimation.

6. (i) (5 points) State Fubini's Theorem.
   (ii) (5 points) State the Dominated Convergence Theorem.
In the remaining part of this question, you may assume without proof that

$$\int_{2\pi}^a e^{-tx}\cos(x)dx = \frac{(\sin(a) - t\cos(a))e^{-ta} + te^{-2\pi t}}{t^2+1}$$

for $t > 0$ and $a > 2\pi$.

(iii) (10 points) By considering the iterated integral $\displaystyle\int_{x=2\pi}^a \cos(x) \int_{t=0}^{\infty} e^{-tx}dt\,dx$, show that

$$\int_{x=2\pi}^{\infty} \frac{\cos(x)}{x}dx = \int_{t=0}^{\infty} \frac{te^{-2\pi t}}{t^2+1}dt$$

where the integral on the left is treated as an improper integral. Be sure to justify all steps.

Solution:

(i) Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be σ-finite measure spaces. If any one of the three quantities

$$\iint |f(x,y)| d\nu(y) d\mu(x) = \int |f| d(\mu \times \nu) = \iint |f(x,y)| d\mu(x) d\nu(y).$$

is finite, then

$$\iint f(x,y) d\nu(y) d\mu(x) = \int f d(\mu \times \nu) = \iint f(x,y) d\mu(x) d\nu(y).$$

(ii) Let f_n be a sequence of measurable functions and suppose that $f_n \longrightarrow f$ pointwise. Further suppose that there is a (nonnegative) function g such that $|f_n| \leq g$ pointwise for every $n \in \mathbb{N}$. If $\int g d\mu < \infty$, then necessarily

$$\int f_n d\mu \underset{n \to \infty}{\longrightarrow} \int f d\mu.$$

(iii) We leave a fixed for the moment. Then

$$\int_{x=2\pi}^{a} |\cos(x)| \int_{t=0}^{\infty} e^{-tx} dt \, dx = \int_{x=2\pi}^{a} \frac{|\cos(x)|}{x} dx < \infty$$

so from Fubini's Theorem we have

$$\int_{x=2\pi}^{a} \frac{\cos(x)}{x} dx = \int_{x=2\pi}^{a} \cos(x) \int_{t=0}^{\infty} e^{-tx} dt \, dx$$

$$= \int_{t=0}^{\infty} \int_{x=2\pi}^{a} \cos(x) e^{-tx} dx \, dt$$

$$= \int_{t=0}^{\infty} \frac{(\sin(a) - t\cos(a))e^{-ta} + te^{-2\pi t}}{t^2 + 1} dt$$

Therefore, bearing in mind that $\displaystyle\int_{x=2\pi}^{\infty} \frac{\cos(x)}{x} dx$ is interpreted as an improper integral (note that it is not defined as a lebesgue integral), it suffices only to show that

$$\int_{t=0}^{\infty} \frac{(\sin(a) - t\cos(a))e^{-ta}}{t^2 + 1} dt \underset{a \to \infty}{\longrightarrow} 0.$$

We could use dominated convergence here, but it is quite easy to use hard estimates

$$\left| \int_{t=0}^{\infty} \frac{(\sin(a) - t\cos(a))e^{-ta}}{t^2 + 1} dt \right| \leq \int_{t=0}^{\infty} \frac{t+1}{t^2+1} e^{-ta} dt$$

$$\leq \int_{t=0}^{\infty} 2e^{-ta} dt = 2a^{-1} \underset{a \to \infty}{\longrightarrow} 0.$$

since $2(t^2 + 1) - (t+1) = 2(t - \frac{1}{4})^2 + \frac{7}{8} \geq 0$.

7. Let f be an L^1 function on the circle group $\mathbb{T}$ for the normalized linear measure η. For each of the following properties of f determine an equivalent condition on the sequence $(\hat{f}(n))_{n \in \mathbb{Z}}$ of Fourier coefficients and briefly justify your answer.

(i) (7 points) f is real-valued.

(ii) (6 points) $f \in L^2(\mathbb{T}, \eta)$.

(iii) (7 points) f has an infinitely differentiable version.

Solution:

(i) We have

$$\overline{f}\,\hat{}\,(n) = \int \overline{f(t)} e^{-int} d\eta(t)$$

$$= \overline{\left[\int f(t) e^{int} d\eta(t) \right]}$$

$$= \overline{\hat{f}(-n)}$$

So, if f is real, then $\overline{f} = f$ and $\hat{f}(n) = \overline{\hat{f}(-n)}$. Conversely, if $\hat{f}(n) = \overline{\hat{f}(-n)}$ then f and $\overline{f}$ have the same Fourier transform and by the uniqueness theorem, they must agree (almost everywhere). Hence f is real (almost everywhere).

(ii) Since the exponential system $e_n(t) = e^{int} (n \in \mathbb{Z})$ is an orthonormal basis for $L^2(\mathbb{T})$, we have $\|f\|_2^2 = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2$. A necessary and sufficient condition for $f \in L^2$ is then $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 < \infty$.

(iii) We have from integration by parts

$$(f')\hat{}\,(n) = \int f'(t) e^{-int} d\eta(t) = in \int f(t) e^{-int} d\eta(t) = in\hat{f}(n)$$

provided that f and f' are continuous. Thus, if f is k times continuously differentiable, we have

$$\left(\left(\frac{d}{dt} \right)^k f \right)\hat{}\,(n) = (in)^k \hat{f}(n)$$

and since the kth derivative is in L^2 this gives $\displaystyle\sum_{n \in \mathbb{Z}} |n|^{2k} |\hat{f}(n)|^2 < \infty$. It follows that for every $k \in \mathbb{N}$, $\sup_{n \in \mathbb{Z}} |n|^k |\hat{f}(n)| < \infty$.

Conversely, if for every $k \in \mathbb{N}$, $\sup_{n \in \mathbb{Z}} |n|^k |\hat{f}(n)| < \infty$, then replacing k with $k+2$ we find $\displaystyle\sum_{n \in \mathbb{Z}} |n|^k |\hat{f}(n)| < \infty$. Consequently all the Fourier series $\displaystyle\sum_{n \in \mathbb{Z}} (in)^\ell \hat{f}(n) e^{int}$ for $\ell = 0, 1, \ldots, k$ converge absolutely and uniformly and standard results from MATH 255 allow one to conclude that $\left(\frac{d}{dt} \right)^k f$ is continuous.

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