McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 355

ANALYSIS 4

Examiner: Professor J. Toth
Associate Examiner: Professor V. Jaksic

Date: Friday April 22 , 2005
Time: 2:00 p.m -5:00 p.m

INSTRUCTIONS

Please answer questions in the exam booklets provided.
Each Question is worth 10 points.
Please show all your work.
No Calculators are permitted.

This exam comprises the cover page, and one page of 5 questions.

Mathematics 355 : Final Examination

- **Each question is worth 10 points.**
- **No calculators are permitted.**
- **Please show all your work.**

1. Let $X = [0,1]$, $\mathcal{B}$ the Borel sets, and μ Lebesgue measure. Show that there exists a Borel-measurable $f \in L^1(X; d\mu)$ with the property that $f \notin L^2(X; d\mu)$.

2. Let $a > 0$ and $f(x)$ be a continuous function on $[-a, a]$. Compute the limit

$$\lim_{N \to \infty} N^{1/2} \int_{-a}^{a} e^{-Nx^2/2} f(x) dx.$$

Justify each step carefully. (Hint: Make a change of variables in the integral).

3. Let $(X, \mathcal{F}, \mu)$ be a measure space, A and B be measurable subsets of X and $S(A, B) := (A - B) \cup (B - A)$. Show that, if $\mu(S(A, B)) = 0$, then, for every nonnegative measurable function f,

$$\int_A f d\mu = \int_B f d\mu.$$

4. Let f and g be in $L^2((-\pi, \pi]; dx)$. Extend them to functions on the real line be requiring that they be periodic of period 2π. Show the *convolution*

$$F(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x - y) dy,$$

is in $L^1(-\pi, \pi)$ and that its Fourier coefficients c_n are just

$$c_n = a_n b_n,$$

where, a_n and b_n are the Fourier coefficients of f and g. Justify each step carefully.

5. Let $\chi_{[-1,1]}$ be the indicator function of the interval $[-1, 1]$ (ie. its characteristic function).
(i) Compute $\widehat{\chi_{[-1,1]}}(y)$.
(ii) Use the Plancherel formula to compute

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x} \right)^2 dx.$$