1. (i) (5 marks) Define the concept σ-*field*.
   (ii) (5 marks) Define the concept *measure* on a σ-field.
   (iii) (10 marks) Let $\mathcal{M}$ be a σ-field on a set X and μ a measure on $(X, \mathcal{M})$. Show that

$$|\mu(A) - \mu(B)| \leq \mu(A \triangle B)$$

provided that either $\mu(A) < \infty$ or $\mu(B) < \infty$.

Note: The symbolism $A \triangle B$ denotes $(A \setminus B) \cup (B \setminus A)$, the symmetric difference of A and B.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space.
   (i) (5 marks) Give the definition of $\int f(x) d\mu(x)$ for a nonnegative $\mathcal{M}$-measurable function f on X in terms of the integral of nonnegative $\mathcal{M}$-measurable simple functions on X.

   Let f be a nonnegative $\mathcal{M}$-measurable function on X satisfying $\int f(x) d\mu(x) < \infty$.

   (ii) (5 marks) For $t > 0$, prove the inequality $\mu(\{x; f(x) > t\}) \leq \dfrac{1}{t} \displaystyle\int f(x) d\mu(x)$.
   (iii) (5 marks) Show that $\lim_{t \to \infty} t\mu(\{x; f(x) > t\}) = 0$.
   (iv) (5 marks) Show that $\lim_{t \to 0+} t\mu(\{x; f(x) > t\}) = 0$.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and f be a nonnegative $\mathcal{M}$-measurable function on X satisfying $\int f(x) d\mu(x) = 1$.
   (i) (5 marks) State the Dominated Convergence Theorem.
   (ii) (5 marks) Find

$$\lim_{n \to \infty} \int n \ln \left(1 + \frac{f(x)}{n}\right) d\mu(x).$$

   (iii) (5 marks) State Fatou's Lemma.
   (iv) (5 marks) Find

$$\lim_{n \to \infty} \int n \ln \left(1 + \sqrt{\frac{f(x)}{n}}\right) d\mu(x).$$

4. (i) (5 marks) State carefully Tonelli's Theorem on the integration of nonnegative functions.
   (ii) (5 marks) Explain how the formula

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_{p,q} = \sum_{q=0}^{\infty}\sum_{p=0}^{\infty} a_{p,q}$$

   can be obtained from Tonelli's Theorem in case $a_{p,q} \geq 0$ for all p and q.
   (iii) (10 marks) Evaluate $\int_0^\infty \left(\frac{\sin(x)}{x}\right)^2 dx$. You may use the formulæ

$$x^{-2} = \int_0^\infty e^{-xt} t dt \quad (\forall x > 0) \quad \text{and} \quad \frac{2}{t(t^2+4)} = \int_0^\infty e^{-xt}(\sin(x))^2 dx \quad (\forall t > 0)$$

   without justification.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space.
   (i) (4 marks) State Hölder's inequality for measurable functions on $(X, \mathcal{M}, \mu)$.
   (ii) (4 marks) State Minkowski's inequality for measurable functions on $(X, \mathcal{M}, \mu)$.
   (iii) (4 marks) Define the space $L^p(X, \mathcal{M}, \mu)$ for $1 \leq p < \infty$ and write down the corresponding norm.
   (iv) (8 marks) Suppose that $f, g \in L^1(X, \mathcal{M}, \mu)$ and that $\|f + g\|_1 = \|f\|_1 + \|g\|_1$. Show that $\overline{f}g \geq 0$ (μ-almost everywhere).

6. Let H be a Hilbert space and M a closed linear subspace of H.
   (i) (3 marks) What is meant by the orthogonal complement of M in H.
   (ii) (2 marks) What is meant by the orthogonal projection operator from H to M.

   Now let $H = L^2(\mathbb{R}, \mathcal{L}, \mu)$ where $\mathcal{L}$ is the Lebesgue σ-field and μ is Lebesgue measure. For each of the following closed subspaces M describe as explicitly as you can the orthogonal projection operator from H to M.

   (iii) (5 marks) $M = \{f; f \in H,\ f \text{ vanishes off } [0,1]\}$.
   (iv) (5 marks) $M = \{f; f \in H,\ f(x) = f(-x),\ \mu-\text{a.a. } x \in \mathbb{R}\}$.
   (v) (5 marks) $M = \{f; f \in H,\ f \text{ is } \mu-\text{a.e. constant on each interval } [n, n+1[, n \in \mathbb{Z}\}$.

7. (i) (5 marks) Write down the formula for the Dirichlet kernel D_N on $\mathbb{T}$.

(ii) (5 marks) How is the Dirichlet kernel related to the Fourier partial sum $S_N f$ of a function f?.

(iii) (5 marks) Show that

$$f(t) - S_N f(t) = \frac{1}{2\pi} \int_0^{2\pi} D_N(s) \Big(f(t) - f(t-s) \Big) ds$$

(iv) (5 marks) Show that there is an absolute constant C (independent of N) such that

$$\|f - S_N f\|_\infty \leq C \|f'\|_\infty$$

for all continuously differentiable functions f on $\mathbb{T}$ and all $N = 1, 2, \ldots$

Note: f' denotes the derivative of f.

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FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-355B

Analysis III

Examiner: Professor S. W. Drury
Associate Examiner: Professor K. N. GowriSankaran

Date: Monday, April 29, 2002
Time: 2:00 P.M. – 5:00 P.M.

INSTRUCTIONS

Attempt six questions for full credit.

This is a closed book examination.
Write your answers in the booklets provided.
All questions are of equal weight, each is alloted 20 marks.

This exam comprises the cover and 3 pages of questions.