Final Examination April 21, 1997 Mathematics 189-355B

$(X, \mathcal{A})$ denotes a measurable space, $(X, \mathcal{A}, \mu)$ a measure space.

MARKS

(10) 1. Let $(E_k)_{k=1}^{\infty}$ be a sequence of elements of $\mathcal{A}$. Suppose $\sum_{k=1}^{\infty} \mu(E_k) < +\infty$ and prove that μ-almost all points of X belong only to a finite number of E_k's.

(10) 2. State and prove the Lebesgue theorem on convergence in measure of an almost everywhere convergent sequence of functions.

(10) 3. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$. Prove that $E = G \backslash e$ where G is a G_δ-set, $m_n(e) = 0$.

(5) 4. Suppose f is a non-negative measurable function on X, $\int_X f d\mu < +\infty (*)$. Prove that

$$\lim_{a \to +\infty} a\mu(\{f > a\}) = 0 \quad (**) \ .$$

Does $(**)$ imply $(*)$?

(10) 5. Suppose $(*)$ holds (see problem 4). Prove that

$$\lim_{p \downarrow 0} \int_X f^p d\mu = \mu(\{f > 0\}) \ .$$

(15) 6. Let $(r_k)_{k=1}^{\infty}$ be an enumeration of all rational points of $[0, 1]$. Put

$$f_p(x) := \sum_{k=1}^{\infty} \frac{1}{2^k |x - r_k|^p} \ (x \in [0, 1], \ p > 0)$$

$(1/0 = +\infty)$. Prove that $\int_0^1 f_p dm_1 < +\infty$ if $p < 1$, $\int_I f_p dm_1 = +\infty$ for any non-degenerate interval $I \subset [0, 1]$ if $p \geq 1$, $f_p(x) < +\infty$ m_1-a.e. in $[0, 1]$ for any p. (Use the inequality $(A_1 + \ldots + A_n)^\alpha \leq A_1^\alpha + A_2^\alpha + \ldots + A_n^\alpha$ for $0 < \alpha < 1, \ n = 1, 2, \ldots$).

(15) 7. Put

$$I_1 = \int_0^1 \left[\int_1^\infty (e^{-xy} - 2e^{-2xy}) dx \right] dy$$
$$I_2 = \int_0^\infty \left[\int_0^1 (e^{-xy} - 2e^{-2xy}) dy \right] dx$$

Prove that all four integrations can be understood in the sense of Lebesgue and give finite results, but $I_1 \neq I_2$. Does it contradict Fubini's theorem?

(10) 8. Let f be a non-negative Lebesgue measurable function on $\mathbb{R}$. Prove that $\Gamma_f := \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R},\ 0 < y < f(x)\}$ is Lebesgue measurable ($\Gamma_f \in \mathcal{A}_2$), and

$$m_2(\Gamma_f) = \int_{\mathbb{R}} f dm_1 \ .$$

(5) 9. Let f be a non-negative Lebesgue measurable function in $\mathbb{R}^2$. Suppose that for m_1-almost all $x \in \mathbb{R}$, f_x is m_1-a.e. finite. Prove that for m_1-almost all $x \in \mathbb{R}$, f^x is m_1-a.e. finite.

(10) 10. Let f, g be summable functions on X. Suppose $\mu(X) = 1$ and prove

$$\int_X f d\mu \cdot \int_X g d\mu \leq \int_X f g d\mu$$

if f and g are comonotone, that is $(f(x) - f(y)) \cdot (g(x) - g(y)) \geq 0$ for any $x, y \in X$.

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-355B

ANALYSIS II (PART II)

Examiner: Professor V. Havin
Associate Examiner: Professor J.R. Choksi

Date: Monday, April 21, 1997
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

Solve all problems

This exam comprises the cover and 2 pages of questions.