McGILL UNIVERSITY
FACULTY OF SCIENCE
FINAL EXAMINATION

MATHEMATICS 189-355B
**Analysis II**

Examiner: Prof. C. Herz
Associate Examiner: Prof. J. Choksi

Tuesday, April 18, 1995
2:00-5:00 PM

*Instructions*: **Answer all 6 questions. No calculators permitted.**

This exam comprises this cover and one page.

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1. Let $(X_n) \subset L^2(\mathcal{M}, \mu)$ be an orthogonal sequence with $\|X_n\|_2 \leq 1$ for all $n$. Prove that $\dfrac{X_1 + \cdots + X_n}{n}$ converges to 0 in measure.

2. Let $\mathcal{M}$ be a $\sigma$-algebra with largest element $X$. Suppose $(\mu_n; n \in \mathbf{N})$ is a sequence of countably-additive measures on $\mathcal{M}$ with $0 \leq \mu_n(E) \leq \mu_{n+1}(E)$ for all $E \in \mathcal{M}$. Put $\mu = \lim_{n\to\infty} \mu_n$ and suppose that $\mu(X) < \infty$. Prove that $\mu$ is a countably-additive measure.

3. Let $(\mathcal{M}, \mu)$ be an arbitrary measure algebra. Assume $1 < p < \infty$ and $p' = p/(p-1)$. Suppose $(f_n) \subset L^p(\mathcal{M}, \mu)$ and $(g_n) \subset L^{p'}(\mathcal{M}, \mu)$. Suppose $f_n \to f$ in the $L^p$-norm and $g_n \to g$ in the $L^{p'}$-norm. Show that $\lim_{n\to\infty} \|f_n g_n - fg\|_1 = 0$ and $\displaystyle\int_X f_n g_n \, d\mu \to \int_X fg \, d\mu$.

4. Find the value of $\displaystyle\lim_{n\to\infty} \int_0^n (1 + x/n)^n e^{-2x} \, dx$ and prove the result.

5. Suppose that $f \in L^1(\mathcal{M}, \mu)$. Prove that given $\epsilon > 0$ there exists $\delta > 0$ such that $\displaystyle\int_E |f| \, d\mu < \epsilon$ whenever $E \in \mathcal{M}$ and $\mu(E) < \delta$.

6. Let $W^t$ be the Gauss-Weierstrass kernel, $W^t(x) = (2\pi t)^{-1/2} \exp(-x^2/2t)$. Assume that $\displaystyle\int_{-\infty}^{\infty} W^t(x) \, dx = 1$ for $t = 1$. Show that this holds for all $t > 0$. Prove that if $f : \mathbf{R} \to \mathbf{R}$ is a continuous function of compact support then $W^t \star f$ converges uniformly to $f$ as $t \to 0$ where $\star$ designates convolution. Prove that if $f \in L^p(\mathbf{R})$, $1 < p < \infty$, then $\lim_{t\to\infty} \|W^t \star f\|_p = 0$.