MCGILL UNIVERSITY
FACULTY OF SCIENCE

Final Examination

MATH 354
HONOURS ANALYSIS 3

Examiner: Professor V. Jaksic
Associate Examiner: Professor S. Drury

Thursday December 13, 2007
Time: 2:00 PM to 5:00 PM

Family Name (Please Print): ______________________

First Name: ________________________

Student ID#: _____________________

INSTRUCTIONS

1. Fill in the above clearly.
2. Do not tear any pages from this book.
3. Write your solutions in a clear, complete and logical way.
4. There are 6 questions worth a total of 80 points. The value of each question is indicated in the margin.
5. This is a closed book examination. No notes, books or calculators are allowed.
6. Use of a regular and or translation dictionary is not permitted.
7. This examination consists 19 pages including this cover page. There are 10 empty pages at the end of this exam. You may use them if you need extra space.

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1. Let (X, d) be a metric space and let A and B be two subsets of X. Prove that

$$\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B).$$

$\overline{10}$ **2.** Let (X, d) be a metric space and let A and B be two connected subsets of X such that $A \cap B \neq \emptyset$. Prove that $A \cup B$ is also a connected subset of X.

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3. Let (X, d) be a connected metric space and let $f : X \mapsto \mathbf{R}$ be a continuous function. Let $a = \inf_{x \in X} f(x)$, $b = \sup_{x \in X} f(x)$. Show that for any $r \in (a, b)$ there exist $x \in X$ such that $f(x) = r$.

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4. Let (X, d) be a compact metric space.

(a) [10 points] Prove that X is complete.

(b) [10 points] Let $f : X \mapsto \mathbf{R}$ be a continuous function. Prove that there exists $x_0 \in X$ such that

$$f(x_0) = \sup_{x \in X} f(x).$$

Additional page for the Problem 4.

5. Let $l_2(\mathbf{N})$ the vector space of all square summable sequences of real numbers $x = (x_n)_{n=1}^{\infty}$ equipped with the norm

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$$\|x\| = \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2}.$$

(a) [10 points] Prove that $l_2(\mathbf{N})$ is complete.

(b) [10 points] Let

$$X = \{x \in l_2(\mathbf{N}) \, : \, |x_n| \leq 1/n \text{ for all } n\}.$$

Prove that X is a compact subset of $l_2(\mathbf{N})$.

Additional page for the Problem 5.

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6. Let k_1 and k_2 be positive constants and let $X \subset C([0,1])$ be the set of all continuously differentiable functions $f : [0,1] \to \mathbf{R}$ satisfying

$$|f(0)| \leq k_1, \qquad \int_0^1 (f'(t))^2 dt \leq k_2.$$

Prove that X is bounded and equicontinuous and deduce that $\mathrm{cl}(X)$ is a compact subset of $C([a,b])$.

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