1. A metric space (X, ρ) is said to be locally compact if for every $x \in X$, there exists an open set U_x, with $x \in U_x$ and $\overline{U}_x$ compact. Prove that (i) $\mathbb{R}^n$ is locally compact, and (ii) ℓ^1 is not locally compact.

2. Let (X, ρ) be a metric space with at least 2 distinct points. Show that there exists a non constant continuous function $X \to \mathbb{R}$. If further X is connected, show that X must be uncountable.

3. (a) Define what is meant by a connected component of a metric space (X, ρ). If $E \subset X$ is non-empty, open, closed and connected, show that E is a component.

   (b) If (X, ρ) is a connected metric space, $f : X \to Y$ is a continuous map, show that $f(X)$ is connected.
   Show that the circle $T = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is not homeomorphic to the closed interval $[0, 1]$.

4. (a) State the Ascoli-Arzela theorem on the relative compactness of sets of continuous functions on a compact metric space.

   (b) Let $\alpha > 0,\ M > 0$ be fixed; let

   $$E = \{f \in C([0,1]) : \ |f(x)| \leq M,\ |f(x) - f(y)| \leq M|x - y|^\alpha,\ \forall x, y \in [0,1]\}.$$

   Show that E is relatively compact in $C[0, 1]$.

   (c) Give an example of a countable infinite set of functions on a closed, bounded interval, which is neither equicontinuous nor uniformly bounded.

5. (a) State the Stone-Weierstrass theorem for real-valued functions on a compact metric space.

   (b) For any bounded closed interval in $\mathbb{R}$, let $\mathcal{P}[a, b]$ denote the real vector space of real-valued polynomials defined on $[a, b]$.
   If $f : [a, b] \times [c, d] \to \mathbb{R}$ is continuous, show that f can be uniformly approximated by functions of the form

   $$g_1(x)h_1(y) + \cdots + g_k(x)h_k(y),$$
   $$k \in \mathbb{N},\ g_j \in \mathcal{P}[a, b],\ h_j \in \mathcal{P}[c, d],\ j = 1, \cdots, k;\ x \in [a, b],\ y \in [c, d].$$

6. (a) State the inverse and implicit function theorems.
   (b) Let $\underline{f}:\mathbb{R}^2 \to \mathbb{R}^2$ be the map $\underline{f}(x,y) = (u,v)$ with $u = x,\ v = xy$. Find $\underline{f}'$, and determine at what points (x,y) the map $\underline{f}$ is locally one to one. Is the map one to one on <u>all</u> of $\mathbb{R}^2$? Find the image under $\underline{f}$ of the rectangle $\{(x,y) : 1 \le x \le 2,\ 0 \le y \le 2\}$.
   (c) Under what conditions do the equations

$$F(x,y,z) = 0, \qquad G(x,y,z) = 0$$

determine x, y as functions $x = f(z),\ y = g(z)$ of z, near a point (x_0, y_0, z_0) which satisfies these two equations? Apply this to the functions

$$F(x,y,z) = z^2 + xy - a, \quad G(x,y,z) = z^2 + x^2 - y^2 - b,$$

where $a, b \in \mathbb{R}$ and are constant. Compute $f'(x)$ and $g'(z)$ for these functions, and a point (x_0, y_0, z_0) which satisfies the equations as well as the conditions you have found.

7. Prove or disprove each of the following: If the statement is true give a proof, if it is false, give a counter example.
   (a) If F is closed, $F \subseteq \mathbb{R},\ F$ uncountable then the interior $F^\circ \ne \emptyset$.
   (b) If X is compact, $f : X \to Y$ is continuous, then $f(X)$ is compact.
   (c) $\{\sin nx : n \in \mathbb{N}\}$ is equicontinuous on $[0, 2\pi]$.

McGILL UNIVERSITY

FACULTY OF SCIENCE

SUPPLEMENTAL/DEFERRED EXAMINATION

MATHEMATICS 189-354A

ANALYSIS III

Examiner: Professor J.R. Choksi
Associate Examiner: Professor S.W. Drury

Date: Tuesday, May 4, 1999
Time: 9:00 A.M. - 12:00 Noon.

INSTRUCTIONS

NO CALCULATORS ARE PERMITTED.
All questions carry equal marks.
Attempt any 6 (SIX) questions.

This exam comprises the cover and 2 pages of questions.