Final Examination Monday, December 19, 1994 Mathematics 189-354A

1. (a) Let (X, d_1) and (Y, d_2) be metric spaces. Define:

   (i) Limit point of a subset S of X;

   (ii) Cauchy sequence in X;

   (iii) Continuous function $f : X \to Y$;

   (iv) Equicontinuous family of functions;

   (v) Complete metric space;

   (vi) Compact metric space.

   (b) Define: Differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$.

2. Let (X, d) be a metric space, (f_n) a sequence of continuous, real valued functions on X.

   (a) If f_n converges uniformly on X to a function f, show that f is continuous.

   (b) If further $x_n \to x$ in X, show that $f_n(x_n) \to f(x)$.

3. State Baire's Theorem.
   Suppose (X, d) is a complete metric space. Let $G_1, G_2, \ldots$ be a sequence of open subsets of X. Suppose, in addition, G_n is dense in X, for each n. Prove that $\bigcap_1^\infty G_n$ is also dense in X.

4. Let (f_n) be a uniformly bounded sequence of functions which are Riemann integrable on $[a, b]$. If $F_n(x) = \int_a^x f_n(t)dt,\ a \leq x \leq b$, prove that there exists a subsequence (F_{n_k}) which converges uniformly on $[a, b]$.

5. (a) Prove that every compact metric space (X, d) is separable.

   (b) Prove that if (X, d) is a compact metric space, then $C(X, \mathbb{R})$ is a separable metric space.
   [Hint: Let $\{x_1, x_2, \ldots\}$ be a subset of X; if $f_n(x) = d(x, x_n)$ for all $x \in X$, then $\{1, f_1, f_2, \ldots\}$ generates an algebra in $C(X, \mathbb{R})$.]

6. (a) State Tietze's Extension theorem.

   (b) (i) Prove that in every infinite metric space there is an infinite sequence (x_k) such that no limit point of the set $\{x_1, x_2, \ldots\}$ is an element of the sequence.

   (ii) Let (X, d) be a compact metric space and suppose that the bounded closed sets of $C(X, \mathbb{R})$ are compact; prove that X consists of a finite number of points.

7. (a) Let f be a bijection from the open set $U \subset \mathbb{R}^n$ onto the open set $V \subset \mathbb{R}^n$.

   (i) If f and f^{-1} are differentiable on U and V respectively, prove that the Jacobian $J_f(x) \neq 0$ for all $x \in U$.

   (ii) If in (i) we do not assume differentiability of f^{-1}, is the conclusion ($J_f(x) \neq 0$ for all $x \in U$) still valid?

   (iii) Let f be differentiable in U and let f^{-1} satisfy a Lipschitz condition on V. Prove that f^{-1} is differentiable on V.

   (b) Let U be an open set in $\mathbb{R}^n$, let $u_0 \in U$ and let $f : U \to \mathbb{R}^n$ be continuous on U and continuously differentiable on $U \backslash \{u_0\}$. If $\lim_{x \to u_o} Df(x) = L$, prove that f is also differentiable at u_0 and $Df(u_0) = L$.

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-354A

ANALYSIS II (PART I)

Examiner: Professor R. Vermes
Associate Examiner: Professor W.O.J. Moser

Date: Monday, December 19, 1994
Time: 2:00 P.M. - 5:00 P.M.

This exam comprises the cover and 2 pages of questions.