Student Name:
Student Id#:

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 327/397

Matrix Numerical Analysis and Honors Matrix Numerical Analysis

Examiner: Professor A. Humphries
Associate Examiner: Professor J-C. Nave

Date: Friday April 15, 2011
Time: 2:00 p.m - 5.00 p.m

INSTRUCTIONS

1. **Students in MATH 327 answer 5 of the first 7 questions. If you answer more than 5 questions, credit will be given for the best 5 answers. Do not answer questions 8 or 9.**
2. **Students in MATH 397 answer 5 of the last 6 questions. If you answer all of these questions, credit will be given for the best 5 answers. Do not answer questions 1, 2 or 3.**
3. Please answer all questions in the exam booklets provided, starting each question on a new page.
4. All questions carry equal weight.
5. This is a closed book exam. Notes and textbooks are not permitted.
6. Translation dictionaries (English-French) are permitted.
7. Calculators, including graphical calculators are permitted.
8. This exam comprises of the cover page and 3 pages of 9 questions.

1. (Do not do this question if you are a (Honours) math397 student)
   Give examples non-zero matrices in $\mathbb{R}^{2\times 2}$ which

   (a) has $\rho(A) = \|A\|_1 = \|A\|_\infty = \|A\|_2$,

   (b) has $\rho(A) < \|A\|_1 < \|A\|_\infty < \|A\|_2$,

   (c) is singular and diagonalizable,

   (d) is non-diagonalizable and non-singular,

   (e) has a Jordan factorisation that changes by a large amount when one entry of A is perturbed a small amount, but whose Schur factorisation only changes a small amount.

2. (Do not do this question if you are a (Honours) math397 student)
   Let

$$A = \begin{pmatrix} 5 & 3 & 0 \\ 3 & 7 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

   (a) Find the Cholesky factorization of A.

   (b) Hence or otherwise, find A^{-1}.

   (c) Define the condition number $\kappa(A)$ for an invertible matrix A. Show that $\kappa(A) \geqslant 1$ in any induced norm.

   (d) Find $\kappa_1(A)$ for the matrix A stated above.

3. (Do not do this question if you are a (Honours) math397 student)

   (a) State the simplest form of the QR-algorithm for finding all the eigenvalues of a square matrix.

   (b) Show that the successive matrices $A^{(k)}$ produced in the algorithm are all unitarily similar to A.

   (c) What is an *upper Hessenberg matrix*? Let H be upper Hessenberg, and let $H = QR$ be its QR factorization. Show that RQ is upper Hessenberg (you may assume that Q is upper Hessenberg).

   (d) How can (c) be used to speed up the QR algorithm?

4. (a) Define the term *Unitary matrix*, and show that if P is unitary then $\|Px\|_2 = \|x\|_2$.

   (b) Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ 0 & 2 \\ 0 & 4 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}. \qquad (Ab)$$

   Find the vector v which defines a suitable Householder reflector, and hence find a unitary matrix Q^* and an upper triangular matrix R such that $Q^*A = R$.

   (c) Explain how the QR factorisation of a matrix $A \in \mathbb{C}^{m\times n}$ may be used to solve the least squares problem of finding $x \in \mathbb{C}^n$ such that

$$\|b - Ax\|_2 = \min_{y\in\mathbb{C}^n} \|b - Ay\|_2, \quad \text{where } b \in \mathbb{C}^m \text{ and } m \geqslant n. \qquad (LLSP)$$

   (d) Solve the linear least squares problem (LLSP) when A and b are defined by (Ab).

5. (a) Define the relative condition number $\kappa(x)$ of the function $f : \mathbb{R} \to \mathbb{R}$.

   (b) Find the relative condition number $\kappa(x)$ of the function $f(x) = \cos(x)$ (with respect to perturbations of x).

   (c) Define the concept of *backward stability.*

   (d) Show that on a computer which satisfies the axioms (P1) and (P2):
   (P1): For all $x \in \mathbb{R}$, there exists ε with $|\varepsilon| \leqslant \varepsilon_{mach}$ and $fl(x) \in \mathbb{F}$ such that $fl(x) = x(1+\varepsilon)$.
   (P2): For every elementary operation $*$ and all $x, y \in \mathbb{F}$, there exists ε with $|\varepsilon| \leqslant \varepsilon_{mach}$ such that $x \circledast y = (x * y)(1+\varepsilon)$.
   that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = x_1 * x_2$ is backward stable with respect to perturbations of $x = (x_1, x_2)$ for every elementary operation $*$.

6. Let

$$A = \begin{pmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{pmatrix}, \qquad b = \begin{pmatrix} 13 \\ -2 \\ 24 \\ -14 \end{pmatrix}.$$

   (a) Find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

   (b) Find the solution x of $Ax = b$.

   (c) For a general matrix $A \in \mathbb{C}^{m \times m}$, why might the algorithm you used in (a) fail? What properties do the matrices P and L have in the generalized decomposition $PA = LU$?

   (d) If the decomposition $PA = LU$ has already been computed, how many additional floating point operations are required to solve $Ax = b$?

7. (a) Let (λ_i, v_i) for $i = 1, \ldots, n$ be the eigenvalues and eigenvectors of A. Let $\mu \in \mathbb{C}$ be such that $(A - \mu I)$ is invertible. What are the eigenvalues and eigenvectors of $(A - \mu I)^{-1}$?

   (b) Show that α equal to the Rayleigh quotient solves the linear least squares problem $\min_\alpha \|Ax - x\alpha\|_2$ when x is not an eigenvector of A.

   (c) Describe the Inverse Power Method with fixed shift for finding the eigenvalue of A which is closest to $\mu \in \mathbb{C}$.

   (d) When does the Inverse Power Method with fixed shift converge, and what is the rate of convergence? (You don't need to justify your answer).

8. (Do not do this question if you are a (Majors) math327 student)

   (a) For $A \in \mathbb{C}^{m\times n}$ define the ∞-norm, $\|A\|_\infty$ as an induced norm.

   (b) Show that $\|A\|_\infty$ defines a norm.

   (c) Show that
   $$\|A\|_\infty = \max_{1\leqslant i\leqslant m} \sum_{j=1}^{n} |a_{ij}|.$$

   (d) Show that for all $v \in \mathbb{C}^n$
   $$\|v\|_\infty \leqslant \|v\|_2, \qquad \text{and} \qquad \|v\|_2 \leqslant \sqrt{n}\|v\|_\infty.$$
   Use this to show that for $A \in \mathbb{C}^{n\times n}$
   $$\frac{1}{\sqrt{n}}\|A\|_\infty \leqslant \|A\|_2 \leqslant \sqrt{n}\|A\|_\infty.$$

9. (Do not do this question if you are a (Majors) math327 student)

   (a) Define the condition number $\kappa(A)$ of an invertible matrix $A \in \mathbb{C}^{m\times m}$.

   (b) Show that A^*A is invertible if and only if $A \in \mathbb{C}^{m\times n}$ with $m \geqslant n$ has full rank.

   (c) Define the pseudo inverse of the full rank matrix $A \in \mathbb{C}^{m\times n}$ with $m \geqslant n$. How is the condition number of a singular full rank matrix $A \in \mathbb{C}^{m\times n}$ with $m \geqslant n$ defined?

   (d) Let $A \in \mathbb{C}^{m\times n}$ with $m \geqslant n$ be of full rank and have singular value decomposition $A = U\Sigma V^*$. Derive expressions for $\kappa_2(A)$ and $\kappa_2(A^*A)$ in terms of the singular values of A, and hence show that $\kappa_2(A^*A) = [\kappa_2(A)]^2$. Briefly, what are the implications for solving linear least squares problems using the normal equations?