McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 326 AND MATH 376

NONLINEAR DYNAMICS AND CHAOS AND
HONOURS NONLINEAR DYNAMICS

Examiner: Professor P. Tupper   Date: Friday December 7, 2007
Associate Examiner: Professor A. Humphries   Time: 2:00 PM - 5:00 PM

INSTRUCTIONS

1. Please answer questions in the exam booklets provided.

2. This is a closed book exam. No notes, textbooks or crib sheets are permitted.

3. Calculators are not permitted.

4. Use of a Translation dictionary is permitted.

**This exam comprises of the cover page, and two pages of eight
questions.**

Math 326/376, Fall 2007, Final Exam
Friday, December 7, 2007

CLOSED BOOK EXAM.
NO CALCULATORS PERMITTED.

**Majors students (326) do any six questions.
Honours students (376) do Questions 3 through 8.**

**All questions are worth the same number of points.
When in doubt, justify your answer.**

1. (Do not do this question if you are a 376 student.) Find and classify the fixed points of the following system, justifying your answer. On one plot sketch the nullclines and the direction of the flow across the nullclines. On another plot sketch a plausible phase portrait. Do not worry about the eigenvectors of the Jacobian at the fixed points.

$$\dot{x} = x^2 - y, \quad \dot{y} = x - y.$$

2. (Do not do this question if you are a 376 student.) Give examples of functions $f(x, \mu)$ such that the system $\dot{x} = f(x, \mu)$ has a bifurcation of the following type at $x = 1, \mu = 0$:

   (a) saddle-node

   (b) transcritical

   (c) supercritical pitchfork

   (d) subcritical pitchfork

   You do not need to justify your answer. **Note:** Make sure the bifurcations occur at $x = 1$.

3. (a) Consider the system on the circle given by

   $$\dot{\theta} = \sin 2\theta.$$

   Identify and classify its fixed points and sketch a phase portrait on the circle.

   (b) Consider the system on the plane given by

   $$\dot{\theta} = \sin 2\theta, \qquad \dot{r} = r - r^3,$$

   where θ and r are polar coordinates. Identify all fixed points and classify them. On one plot draw the nullclines of the system and show the direction of the flow across them. On another plot sketch a phase portrait. Each stable fixed point in the system has a basin of attraction. In a third plot indicate the basin of attraction of each stable fixed point.

See next page for more questions.

4. Consider the system $\dot{x} = \mu + x - x^3$, where μ is a parameter.

   (a) Identify any values of μ at which bifurcations occur. Classify the bifurcations.

   (b) Sketch a bifurcation diagram for the fixed points of the system versus μ.

   (c) Suppose you have a physical system with a variable x that you can observe and a parameter μ which you can control. Suppose that the equation above is a good model for this system. Imagine starting the system with $\mu = -5$ and slowing increasing μ up to $\mu = 5$. Sketch a plot showing how you think x would change with respect to μ during this procedure.

5. Consider the system $\ddot{x} = -\sin x$ for x on the real line.

   (a) Rewrite it as a first-order system for variables x and y.

   (b) Find and classify the fixed points, justifying your answer.

   (c) Sketch the phase portrait.

   (d) Find an equation for a heteroclinic orbit in the system. (A heteroclinic orbit is a trajectory of the system that links to fixed points.)

6. Give examples of parametrized systems of differential equations that describe flows in the plane with the following bifurcations. In each case identify the value of the parameter at which the bifurcation occurs and describe what happens to the relevant fixed points and cycles in the system. I recommend that you use polar coordinates. You do not need to justify your answer.

   (a) infinite period bifurcation

   (b) saddle-node bifurcation of cycles

   (c) subcritical Hopf bifurcation

   (d) supercritical Hopf bifurcation

7. Consider the system
$$\dot{x} = x(\mu - x - y), \qquad \dot{y} = y(1 - x - y)$$
   where $\mu < 1$ is a parameter.

   (a) Find all the fixed points for $x \geq 0, y \geq 0$ and their dependence on $\mu < 1$, and classify them.

   (b) At what value of $\mu < 1$ does a bifurcation occur? Describe the qualitative change that occur at this bifurcation.

   (c) Sketch two plausible phase portraits for $x \geq 0, y \geq 0$, one for μ on each side of the bifurcation point.

8. Show that the following system has a periodic orbit. Cite any results you use from class or the text. The variables r and θ denote polar coordinates.

$$\dot{r} = r(4 - r^2) + r \sin 2\theta,$$
$$\dot{\theta} = 1.$$