McGill University April 2012
Faculty of Science Final examination

Honours Ordinary Differential Equations

Math 325

Friday, April 27th, 2012

Time: 2pm-5pm

Examiner: Prof. A.R. Humphries Associate Examiner: Prof. N. Sancho

INSTRUCTIONS

1. Answer all questions in the exam booklets provided. Start each question on a new page.
2. All questions carry equal weight.
3. This is a closed book exam. No crib sheets, textbooks or any other aids are permitted.
4. Calculators are permitted.
5. Use of a regular dictionary is not permitted.
6. Use of a translation dictionary is permitted.

This exam comprises the cover page, 2 pages of 6 questions and a table of Laplace Transforms.

1.) (a) Find (in explicit form) the solution $y(x)$ of the initial value problem

$$yy' = 1 - 2x, \quad y(1) = -2.$$

What is the interval of validity of the solution?

(b) Find the general solution $y(x)$ of the fourth order equation

$$y^{(4)} - y = 2e^x.$$

2.) Consider the initial value problem

$$y'(x) = f(x, y) = 1 - xy(x), \qquad y(0) = 0,$$

(a) Let $y_0(x) = 0$ for all $x \in \mathbb{R}$, and find the first two approximations $y_1(x)$, $y_2(x)$, to the exact solution $y(x)$ by Picard iteration.

(b) Find an approximation to $y(1)$ using the forward Euler method $y_{n+1} = y_n + hf(x_n, y_n)$ with $h = 0.5$.

(c) Suppose that the solution can be written as $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Find and state a_0 and a_1 and the recurrence relation used to define general a_n. Hence show that $a_{2n} = 0$ for all $n \geqslant 1$. Use the ratio test to show that the solution has infinite radius of convergence. (You do *not* have to state an explicit formula for the coefficient a_{2n+1}.)

3.) (a) Define the Wronskian $W(f_1, f_2, \ldots, f_n)(x)$ of the functions $f_i(x)$ for $i = 1, \ldots, n$.

(b) Show that $n - 1$ times continuously differentiable functions $f_i(x)$ for $i = 1, \ldots, n$ are linearly independent on any interval I containing x_0 if $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$.

(c) State a first order differential equation which is satisfied by $W(y_1, y_2, \ldots, y_n)(x)$ whenever $y_1, \ldots, y_n$ are solutions of the nth order differential equation

$$y^{(n)} + \sum_{j=1}^{n} p_j(x) y^{(n-j)} = 0.$$

In the case $n = 2$, show that $W(y_1, y_2)(x)$ satisfies the first order differential equation that you stated.

(d) Give an example of two functions $y_1(x)$ and $y_2(x)$ which are continuously differentiable and linearly independent on the interval $[-1, 1]$, but for which $W(y_1, y_2)(x) = 0$ for all $x \in [-1, 1]$. Are $y_1(x)$ and $y_2(x)$ a fundamental set of solutions for any linear differential on the interval $[-1, 1]$?

4.) Let

$$L[y] = xy'' - (1 + x)y' + y.$$

(a) Show that $y_1(x) = e^x$ solves the homogeneous problem $L[y_1](x) = 0$.

(b) Find a second solution $y_2(x)$ which solves $L[y_2](x) = 0$ (with y_2 linearly independent of y_1).

(c) Let y_1, y_2 be a fundamental set of solutions to $L[y](x) = 0$. State (but do not derive) the equations which define a particular solution $y_p(x)$ which solves $L[y_p](x) = g(x)$ when using Variation of Parameters.

(d) Find the general solution of

$$L[y](x) = x^2 e^{2x},$$

where $L[y]$ is the differential operator defined above.

5.) Consider the differential equation

$$2x^2y'' - xy' + (1+x)y = 0.$$

(a) Define *regular singular point.* Find the regular singular point of the given equation, state the indicial equation and find its roots.

(b) Find the recurrence relations that define the coefficients of a fundamental set of Frobenius series solutions $y_1(x)$, $y_2(x)$ for $x > x_0$ expanded about the regular singular point x_0. Letting the first coefficient be 1, find the next three coefficients in each series, and hence state expressions for $y_1(x)$, $y_2(x)$ (You do *not* need to find an expression for the nth coefficient in each series solution; your expressions will include a sum over terms including the undetermined coefficients).

(c) Let $y(x) = c_1y_1(x) + c_2y_2(x)$, where $y_1(x)$ and $y_2(x)$ were found in (b). (With limits from the right-hand side so $x > x_0$), what are the possible values of $\lim_{x\to x_0} y(x)$ and $\lim_{x\to x_0} y'(x)$?

6.) (a) State the definition of the Laplace transform $G(s)$ of the function $g(t)$. Let $\mathcal{U}(t)$ be the Heaviside function, and use the definition of the Laplace transform to show directly that $\mathcal{L}\{\mathcal{U}(t-a)g(t-a)\} = e^{-as}G(s)$ when $a > 0$.

(b) Let $y(t)$ solve

$$y'' + 3y' + 2y = \delta(t-\pi) + \mathcal{U}(t-2\pi), \qquad y(0) = 1, \quad y'(0) = -1.$$

Find an expression for $Y(s)$, the Laplace transform of $y(t)$.

(c) Find $y(t)$, the solution of the initial value problem stated in (b).

Table of Laplace Transforms

function $f(t)$	Laplace transform $F(s)$
1	$1/s \quad (s>0)$
t^n	$n!/s^{n+1} \quad (s>0)$
e^{at}	$1/(s-a) \quad (s>a)$
$\sin at$	$a/(s^2+a^2) \quad (s>0)$
$\cos at$	$s/(s^2+a^2) \quad (s>0)$
$\sinh at$	$a/(s^2-a^2) \quad (s>\|a\|)$
$\cosh at$	$s/(s^2-a^2) \quad (s>\|a\|)$
$e^{-at}f(t)$	$F(s+a)$
$\mathcal{U}(t-a)$ or $\mathcal{U}_a(t) \ (a \geqslant 0)$	$e^{-as}/s \quad (s>0)$
$\delta(t-a) \quad (a>0)$	e^{-as}
$\mathcal{U}(t-a)f(t-a)$ or $\mathcal{U}_a(t)f(t-a)$	$e^{-as}F(s)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - f^{(n-1)}(0)$
$(-t)^nf(t)$	$F^{(n)}(s)$
$f * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$