Mathematics 189-325B

Final Examination

April 26, 2000

1. Solve the following differential equations and determine that solution satisfying the initial condition if one is given:

   (a) $x\,y' - 3\,y = x^4, \qquad y(1) = 3\,;$

   (b) $x^2 y' = y^2 - 2x^2, \qquad y(1) = 0;$

   (c) $y' = (x + y - 2)^2\,.$

2. Consider the nonhomogeneous second-order differential equation

$$(1-x)\,y'' + x\,y' - y = 4\,(x-1)^2 e^{-x}\ .$$

   (a) Given that $y_1 = e^x$ is a solution of the homogeneous equation, use the method of reduction of order to find that $y_2 = x$ is a second homogeneous solution.

   (b) Now, obtain the general solution of the original nonhomogeneous ODE. [Useful integral: $\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax-1)$.]

3. Find the general solution in terms of real valued functions of x for

$$y^{iv} + 3\,y'' - 4\,y = 5\,e^x\ .$$

4. Find a solution of the nonlinear ODE

$$2\,y\,y'' = 1 + (y')^2$$

   satisfying the initial conditions $y(0) = y'(0) = 1\,.$

5. Determine all real eigenvalues and eigenfunctions for the boundary-value problem

$$x^2 y'' + x\,y' + 4\,\lambda^2 y = 0, \qquad y(1) = 0 \ \text{ and } \ y'(e^\pi) = 0\,.$$

6. Consider the problem of finding series solutions in powers of x for the equation

$$2x\,y'' + (3 - x\,)y' + (\gamma - 1)y = 0\,,$$

where γ is a constant.

(a) Determine the indicial equation and its roots;

(b) Find the recurrence relation for successive terms in the series;

(c) Show that polynomial solutions exist when γ is an integer and write down such a solution for the case $\gamma = 3$;

(d) A second linearly independent solution $y_2(x)$, say, is in the form of an infinite series. Write down the general solution for the case $\gamma = 3$ including at least the first two terms of $y_2(x)$.

7. (a) Solve the integral equation $y(t) - 2\int_0^t e^{t-\tau} y(\tau)\, d\tau = e^{2t}$ by taking the Laplace transform of both sides.

(b) Use the Laplace transform to solve the initial-value problem

$$y'' - 2y' + 5y = \delta(t - \pi)\cos t,\ y(0) = y'(0) = 1\,.$$