1. (a) Solve the initial-value problem

$$y' - 2xy = 4x, \quad y(0) = 1.$$

(b) Find the solution of

$$3x^2y\,dx + (3x^3 + 5y^2)dy = 0.$$

(c) Find the solution of the Bernoulli equation

$$x^2y' + 2xy = y^3, \text{ satisfying } y(1) = 1.$$

2. (a) Solve the initial-value problem

$$(y+1)y'' = (y')^2, \quad y(0) = 1,\ y'(0) = 2.$$

(b) Given that $y_1(x) = e^{x^2}$ is a solution of

$$y'' - 4xy' + (4x^2 - 2)y = 0,$$

find a second, linearly independent, solution.

3. (a) Find the general solution in terms of real valued functions of x for

$$y''' + 8y = 1 + e^x.$$

(b) Find the general solution to the following equation:

$$y'' + 4y = \sec 2x, \quad -\frac{\pi}{4} < x < \frac{\pi}{4}.$$

4. Consider the problem of finding series solutions in powers of x for the equation

$$(1 - x^2)y'' - 3xy' + \alpha(\alpha + 2)y = 0,$$

where α is a constant.

(a) For general α, find the recurrence relation between the coefficients of your series. In the case $\alpha = 2$, find a polynomial solution satisfying the initial conditions $y(0) = 1$ and $y'(0) = 0$.

(b) When α is not an integer, the series expansions no longer terminate. In that case, for what values of x would you expect the series to converge? Explain why.

(c) If you wanted to determine a series solution in powers of $(x - 1)$, what would you expect the radius of convergence to be? Explain why.

Note: In parts (b) and (c), you do not need to explicitly find the series solution.

5. (a) If $y'(t) = \sin t + \int_0^t y(t-\tau)\cos\tau \, d\tau$ and $y(0) = 0$, find $y(t)$ by taking the Laplace transform of both sides of the equation.

   (b) Use a Laplace transform to solve the initial-value problem

$$y'' - y = 2\delta(t-1), \;\; y(0) = 1, \;\; y'(0) = 0.$$

6. We wish to find series solutions in powers of x for the differential equation

$$2xy'' + 3y' - xy = 0.$$

Determine:

(a) The nature of the point $x = 0$.

(b) The indicial equation and its roots.

(c) The recurrence formula.

(d) The general solution including the first three nonzero terms of each series.

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-325A

ORDINARY DIFFERENTIAL EQUATIONS

Examiner: Professor S.A. Maslowe
Associate Examiner: Professor N.G.F. Sancho

Date: Thursday, December 9, 1999
Time: 9:00 A.M. - 12:00 Noon.

INSTRUCTIONS

Calculators are not permitted.
A table of Laplace transforms is appended.

This exam comprises the cover and two pages of questions, plus a page of tables.