

STUDENT NAME:

STUDENT ID#

McGILL UNIVERSITY  
FACULTY OF SCIENCE

FINAL EXAMINATION

MATH324A

STATISTICS

Examiner: Professor M. Asgharian

Date: Monday, December 19, 2005

Associate Examiner: Professor R. Steele

Time: 9:00 A.M - 12:00 P.M.

INSTRUCTIONS

**Answer ONLY 8 questions.**

**Calculators are permitted.**

**Answer directly on exam.**

**Dictionaries are allowed**

**One page of formulas is permitted.**

Questions	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	

This exam comprises the cover, 9 pages of questions and 5 pages of tables.

1. If  $Y$  has a binomial distribution with parameters  $n$  and  $p$ , then  $\hat{p} = Y/n$  is an unbiased estimator of  $p$ . Another estimator of  $p$  is  $\hat{p}_2 = (Y + 1)/(n + 2)$ .

(a) Derive the bias of  $\hat{p}_2$ . (3 marks)

(b) Derive  $MSE(\hat{p}_1)$  and  $MSE(\hat{p}_2)$ . (4 marks)

(c) For what values of  $p$  is  $MSE(\hat{p}_1) < MSE(\hat{p}_2)$ ? (3 marks)

2. Suppose  $X_1, X_2, \dots, X_n$  be a random sample of size  $n$  from Bernoulli( $p$ ).  
Let  $Y = \sum_{i=1}^n X_i$ .

(a) Find  $\mathbb{E}(\frac{Y}{n})$  and  $\text{Var}(\frac{Y}{n})$ . (4 marks)

(b) Using Chebyshev's inequality show that  $Y/n$  is a consistent estimator of  $p$ . (3 marks)

(c) Let  $\hat{p}_n = Y/n$ . Show that  $(\hat{p}_n - p)/\sqrt{\hat{p}_n(1 - \hat{p}_n)/n}$  converges to a standard normal distribution. (3 marks)

3. Let  $Y_1, Y_2, \dots, Y_n$  denote a random sample from the density function given by

$$f(y; \theta) = \begin{cases} (\frac{1}{\theta})^r y^{r-1} e^{-y^r/\theta}, & \theta > 0, y > 0 \\ 0, & \text{elsewhere} \end{cases}$$

where  $r$  is a known positive constant.

- (a) Find a sufficient statistics for  $\theta$ . (3 marks)
- (b) Find the maximum likelihood estimator of  $\theta$ . (3 marks)
- (c) Is the estimator in part (b) an MVUE for  $\theta$ ? (4 marks)

4. A two-stage clinical trial is planned for testing  $H_0 : p = 0.10$  versus  $H_A : p > 0.10$ , where  $p$  is the proportion of responders among patients who were treated by the protocol treatment. At the first stage, 15 patients are accrued and treated. If 4 or more responders are observed among the (first) 15 patients,  $H_0$  is rejected, the study is terminated, and no more patients are accrued. Otherwise, another 15 patients will be accrued and treated in the second stage. If a total of 6 or more responders are observed among the 30 patients accrued in the two stages (15 in the first stage and 15 more in the second stage), then  $H_0$  is rejected.

(a) Use the binomial table to find the numerical value of  $\alpha$  for this testing procedure. (3 marks)

(b) Use the binomial table to find the probability of rejecting the null hypothesis when using this rejection region if  $p = 0.3$ . (3 marks)

(c) For the rejection region defined above, find  $\beta$  if  $p = 0.3$ . (4 marks)

5. Let  $Y_1, Y_2, \dots, Y_n$  denote a random sample from a population having a Poisson distribution with mean  $\lambda$ .
- (a) Find the form of the rejection region for a most powerful test of  $H_0 : \lambda = \lambda_0$  against  $H_A : \lambda = \lambda_A$ , where  $\lambda_A > \lambda_0$ . (5 marks)
- (b) Recall that  $\sum_{i=1}^n Y_i$  has a Poisson distribution with mean  $n\lambda$ . Indicate how this information can be used to find any constants associated with the rejection region derived in (a). (5 marks)

6. suppose that  $Y_1, Y_2, \dots, Y_n$  constitute a random sample from a normal distribution with unknown mean  $\mu$  and unknown variance  $\sigma^2$ . Find the likelihood ratio test for testing  $H_0 : \mu = \mu_0$  against  $H_A : \mu \neq \mu_0$ . Show that the likelihood ratio test in this case reduces to a t-test. (10 marks)

7. The octane number  $Y$  of refined petroleum is related to the temperature  $x$  of the refining process. An experiment with a small particle catalyst gave a fitted least squares line of  $\hat{y} = 9.36 + 0.155x$  with  $n = 31$ ,  $\bar{x} = 12$ ,  $V(b) = (0.02)^2$ ,  $SS_{Res} = 2.04$  and  $SS_{Reg} = 4.3$ .
- (a) Test the hypotheses that the slope is significantly different from zero at the significance level of 0.05. (3 marks)
- (b) Find a 95% confidence interval for the mean response at  $x = 10$ . (3 marks)
- (c) Find the coefficient of determination,  $R^2$ , and comment on the quality of the fit. (4 marks)



8. The Florida Game and Fish Commission desires to compare the amounts of residue from three chemicals found in the brain tissue of brown pelicans. Independent random samples of ten pelicans each yielded the accompanying results (measurements in parts per million). Is there evidence of sufficient differences among the mean residue amounts, at the 5% level of significance? (10 marks)

Statistic	Chemical		
	DDE	DDD	DDT
Mean	.032	.022	.041
Standard Devision	.014	.008	.017

9. A study was conducted by Joseph Jacobson and Diane Wille to determine the effect of early child care on infant-mother attachment patterns. In the study, 93 infants were classified as either “secure” or “anxious” using the Ainsworth strange-situation paradigm. In addition, the infants were classified according to the average number of hours per week that they spend in child care. The data appear in the accompanying table. (10 marks)

Attachment Pattern	Hours in Child Care		
	Low (0-3 hours)	Moderate (4-19 hours)	High (20-54 hours)
Secure	24	35	5
Anxious	11	10	8

do the data indicate a dependence between attachment patterns and the number of hours spent in child care? Test using  $\alpha = .05$ .

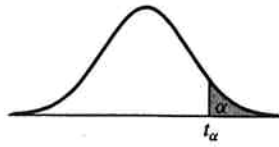
Table 1. (Continued)

(c)  $n = 15$

<i>a</i>	<i>p</i>												<i>a</i>	
	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95		0.99
0	.860	.463	.206	.035	.005	.000	.000	.000	.000	.000	.000	.000	.000	0
1	.990	.829	.549	.167	.035	.005	.000	.000	.000	.000	.000	.000	.000	1
2	1.000	.964	.816	.398	.127	.027	.004	.000	.000	.000	.000	.000	.000	2
3	1.000	.995	.944	.648	.297	.091	.018	.002	.000	.000	.000	.000	.000	3
4	1.000	.999	.987	.836	.515	.217	.059	.009	.001	.000	.000	.000	.000	4
5	1.000	1.000	.998	.939	.722	.403	.151	.034	.004	.000	.000	.000	.000	5
6	1.000	1.000	1.000	.982	.869	.610	.304	.095	.015	.001	.000	.000	.000	6
7	1.000	1.000	1.000	.996	.950	.787	.500	.213	.050	.004	.000	.000	.000	7
8	1.000	1.000	1.000	.999	.985	.905	.696	.390	.131	.018	.000	.000	.000	8
9	1.000	1.000	1.000	1.000	.996	.966	.849	.597	.278	.061	.002	.000	.000	9
10	1.000	1.000	1.000	1.000	.999	.991	.941	.783	.485	.164	.013	.001	.000	10
11	1.000	1.000	1.000	1.000	1.000	.998	.982	.909	.703	.352	.056	.005	.000	11
12	1.000	1.000	1.000	1.000	1.000	1.000	.996	.973	.873	.602	.184	.036	.000	12
13	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.995	.965	.833	.451	.171	.010	13
14	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.995	.965	.794	.537	.140	14

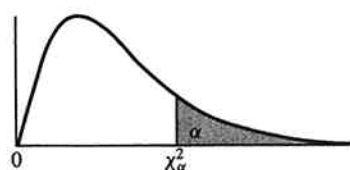
(d)  $n = 20$

<i>a</i>	<i>p</i>													<i>a</i>
	0.01	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	0.99	
0	.818	.358	.122	.012	.001	.000	.000	.000	.000	.000	.000	.000	.000	0
1	.983	.736	.392	.069	.008	.001	.000	.000	.000	.000	.000	.000	.000	1
2	.999	.925	.677	.206	.035	.004	.000	.000	.000	.000	.000	.000	.000	2
3	1.000	.984	.867	.411	.107	.016	.001	.000	.000	.000	.000	.000	.000	3
4	1.000	.997	.957	.630	.238	.051	.006	.000	.000	.000	.000	.000	.000	4
5	1.000	1.000	.989	.804	.416	.126	.021	.002	.000	.000	.000	.000	.000	5
6	1.000	1.000	.998	.913	.608	.250	.058	.006	.000	.000	.000	.000	.000	6
7	1.000	1.000	1.000	.968	.772	.416	.132	.021	.001	.000	.000	.000	.000	7
8	1.000	1.000	1.000	.990	.887	.596	.252	.057	.005	.000	.000	.000	.000	8
9	1.000	1.000	1.000	.997	.952	.755	.412	.128	.017	.001	.000	.000	.000	9
10	1.000	1.000	1.000	.999	.983	.872	.588	.245	.048	.003	.000	.000	.000	10
11	1.000	1.000	1.000	1.000	.995	.943	.748	.404	.113	.010	.000	.000	.000	11
12	1.000	1.000	1.000	1.000	.999	.979	.868	.584	.228	.032	.000	.000	.000	12
13	1.000	1.000	1.000	1.000	1.000	.994	.942	.750	.392	.087	.002	.000	.000	13
14	1.000	1.000	1.000	1.000	1.000	.998	.979	.874	.584	.196	.011	.000	.000	14
15	1.000	1.000	1.000	1.000	1.000	1.000	.994	.949	.762	.370	.043	.003	.000	15
16	1.000	1.000	1.000	1.000	1.000	1.000	.999	.984	.893	.589	.133	.016	.000	16
17	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.996	.965	.794	.323	.075	.001	17
18	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.992	.931	.608	.264	.017	18
19	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.988	.878	.642	.182	19

Table 5. Percentage points of the  $t$  distributions

$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$	d.f.
3.078	6.314	12.706	31.821	63.657	1
1.886	2.920	4.303	6.965	9.925	2
1.638	2.353	3.182	4.541	5.841	3
1.533	2.132	2.776	3.747	4.604	4
1.476	2.015	2.571	3.365	4.032	5
1.440	1.943	2.447	3.143	3.707	6
1.415	1.895	2.365	2.998	3.499	7
1.397	1.860	2.306	2.896	3.355	8
1.383	1.833	2.262	2.821	3.250	9
1.372	1.812	2.228	2.764	3.169	10
1.363	1.796	2.201	2.718	3.106	11
1.356	1.782	2.179	2.681	3.055	12
1.350	1.771	2.160	2.650	3.012	13
1.345	1.761	2.145	2.624	2.977	14
1.341	1.753	2.131	2.602	2.947	15
1.337	1.746	2.120	2.583	2.921	16
1.333	1.740	2.110	2.567	2.898	17
1.330	1.734	2.101	2.552	2.878	18
1.328	1.729	2.093	2.539	2.861	19
1.325	1.725	2.086	2.528	2.845	20
1.323	1.721	2.080	2.518	2.831	21
1.321	1.717	2.074	2.508	2.819	22
1.319	1.714	2.069	2.500	2.807	23
1.318	1.711	2.064	2.492	2.797	24
1.316	1.708	2.060	2.485	2.787	25
1.315	1.706	2.056	2.479	2.779	26
1.314	1.703	2.052	2.473	2.771	27
1.313	1.701	2.048	2.467	2.763	28
1.311	1.699	2.045	2.462	2.756	29
1.282	1.645	1.960	2.326	2.576	inf.

From "Table of Percentage Points of the  $t$ -Distribution." Computed by Maxine Merrington, *Biometrika*, Vol. 32 (1941), p. 300. Reproduced by permission of Professor E. S. Pearson.

Table 6. Percentage points of the  $\chi^2$  distributions

d.f.	$\chi^2_{0.995}$	$\chi^2_{0.990}$	$\chi^2_{0.975}$	$\chi^2_{0.950}$	$\chi^2_{0.900}$
1	0.0000393	0.0001571	0.0009821	0.0039321	0.0157908
2	0.0100251	0.0201007	0.0506356	0.102587	0.210720
3	0.0717212	0.114832	0.215795	0.351846	0.584375
4	0.206990	0.297110	0.484419	0.710721	1.063623
5	0.411740	0.554300	0.831211	1.145476	1.61031
6	0.675727	0.872085	1.237347	1.63539	2.20413
7	0.989265	1.239043	1.68987	2.16735	2.83311
8	1.344419	1.646482	2.17973	2.73264	3.48954
9	1.734926	2.087912	2.70039	3.32511	4.16816
10	2.15585	2.55821	3.24697	3.94030	4.86518
11	2.60321	3.05347	3.81575	4.57481	5.57779
12	3.07382	3.57056	4.40379	5.22603	6.30380
13	3.56503	4.10691	5.00874	5.89186	7.04150
14	4.07468	4.66043	5.62872	6.57063	7.78953
15	4.60094	5.22935	6.26214	7.26094	8.54675
16	5.14224	5.81221	6.90766	7.96164	9.31223
17	5.69724	6.40776	7.56418	8.67176	10.0852
18	6.26481	7.01491	8.23075	9.39046	10.8649
19	6.84398	7.63273	8.90655	10.1170	11.6509
20	7.43386	8.26040	9.59083	10.8508	12.4426
21	8.03366	8.89720	10.28293	11.5913	13.2396
22	8.64272	9.54249	10.9823	12.3380	14.0415
23	9.26042	10.19567	11.6885	13.0905	14.8479
24	9.88623	10.8564	12.4011	13.8484	15.6587
25	10.5197	11.5240	13.1197	14.6114	16.4734
26	11.1603	12.1981	13.8439	15.3791	17.2919
27	11.8076	12.8786	14.5733	16.1513	18.1138
28	12.4613	13.5648	15.3079	16.9279	18.9392
29	13.1211	14.2565	16.0471	17.7083	19.7677
30	13.7867	14.9535	16.7908	18.4926	20.5992
40	20.7065	22.1643	24.4331	26.5093	29.0505
50	27.9907	29.7067	32.3574	34.7642	37.6886
60	35.5346	37.4848	40.4817	43.1879	46.4589
70	43.2752	45.4418	48.7576	51.7393	55.3290
80	51.1720	53.5400	57.1532	60.3915	64.2778
90	59.1963	61.7541	65.6466	69.1260	73.2912
100	67.3276	70.0648	74.2219	77.9295	82.3581

Table 7. (Continued)

Denominator d.f.	$F_{\alpha}$									
	$\alpha$	1	2	3	4	5	6	7	8	9
29	.100	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86
	.050	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22
	.025	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59
	.010	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09
	.005	9.23	6.40	5.28	4.66	4.26	3.98	3.77	3.61	3.48
30	.100	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85
	.050	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
	.025	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57
	.010	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07
	.005	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45
40	.100	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79
	.050	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
	.025	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45
	.010	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
	.005	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22
60	.100	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74
	.050	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
	.025	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33
	.010	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
	.005	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01
120	.100	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68
	.050	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96
	.025	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22
	.010	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56
	.005	8.18	5.54	4.50	3.92	3.55	3.28	3.09	2.93	2.81
$\infty$	.100	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63
	.050	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88
	.025	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11
	.010	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41
	.005	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.74	2.62

From "Tables of percentage points of the inverted beta ( $F$ ) distribution." *Biometrika*, Vol. 33 (1943) by M. Merrington and C. M. Thompson and from Table 18 of *Biometrika Tables for Statisticians*, Vol. 1, Cambridge University Press, 1954, edited by E. S. Pearson and H. O. Hartley. Reproduced with permission of the authors, editors, and *Biometrika* trustees.

Table 7. (Continued)

Denominator d.f.	$F_{\alpha}$									
	$\alpha$	1	2	3	4	5	6	7	8	9
22	.100	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93
	.050	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
	.025	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76
	.010	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
	.005	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81
23	.100	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92
	.050	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32
	.025	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73
	.010	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30
	.005	9.63	6.73	5.58	4.95	4.54	4.26	4.05	3.88	3.75
24	.100	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91
	.050	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
	.025	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70
	.010	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26
	.005	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69
25	.100	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89
	.050	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28
	.025	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68
	.010	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22
	.005	9.48	6.60	5.46	4.84	4.43	4.15	3.94	3.78	3.64
26	.100	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88
	.050	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
	.025	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65
	.010	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18
	.005	9.41	6.54	5.41	4.79	4.38	4.10	3.89	3.73	3.60
27	.100	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87
	.050	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25
	.025	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63
	.010	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15
	.005	9.34	6.49	5.36	4.74	4.34	4.06	3.85	3.69	3.56
28	.100	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87
	.050	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
	.025	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61
	.010	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12
	.005	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52