McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS AND STATISTICS MATH 324

STATISTICS

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Date: 17 December 2004
Time: 09:00 12:00

INSTRUCTIONS

- **Attempt all questions.**
- **Answer in the exam booklet(s) supplied.**
- **This exam will be marked out of 90, out of a possibility of 100 marks.**
- **The function** $\log x$ **is understood to mean** $\log_e(x) = \ln(x)$.
- **The expression** $\mathbb{1}$[*statement*] **takes on value 1 if** *statement* **is true and 0 if** *statement* **is false.**
- **You are not allowed any notes, textbooks or similar material.**
- **Non-programmable calculators are allowed.**
- **Good luck!**

This exam comprises the cover, 5 pages of questions; 3 pages listing pmf's, pdf's and some of their properties; and 2 pages with tables of common sampling distributions.

1. [12 marks total]
Let X and Y have joint probability density function

$$f_{XY}(x,y) = \frac{2}{\theta^2}\mathbb{1}[0 < x, 0 < y, x + y < \theta],$$

and corresponding joint cumulative density function $F_{XY}(x,y)$.

(a) [3 marks]
Show that $f_{XY}(x,y)$ is indeed a density.

(b) **i.** [1 mark]
For which values of (x,y) is $F_{XY} = 0$? You do not need to justify your answer.

ii. [1 mark]
For which values of (x,y) is $F_{XY} = 1$? You do not need to justify your answer.

iii. [2 marks]
Set up an integral to determine $F_{XY}(x,y)$ if $0 < x < \theta$ and $\theta - x \leq y < \theta$. Do not solve the integral.

(c) [2 marks]
Show that $f_X(x) = \frac{2}{\theta^2}(\theta - x)\mathbb{1}[0 < x < \theta]$.

(d) [3 marks]
Let $\theta = 1$, and find the value of $F_{Y|X}(1/2|1/3) = \mathrm{P}[Y \leq 1/2|X = 1/3]$.

2. [10 marks total]
Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, 1)$ distribution (see page 7). Let $\hat{\mu} = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Assume it known that $\hat{\mu}$ is μ-unbiased.

Show that $\hat{\mu}$ is the best (minimum variance) unbiased estimator of μ in two ways:

(a) [6 marks]
by showing that its variance meets the Cramér-Rao lower bound;

(b) [4 marks]
by using the equality condition of the Cramér-Rao theorem.

3. [10 marks total]
Let $Y_1, \ldots, Y_n$ be a random sample from an $\mathrm{Exp}(\beta)$ distribution (see page 6).

(a) Let $Y_{\min} = \min_i Y_i$.

i. [2 marks] Show that $Y_{\min} \sim \mathrm{Exp}(\beta/n)$. (State without proof any formula that you may want to use to prove this result.)

ii. [1 mark] Show that $\tilde{\beta} = nY_{\min}$ is an unbiased estimator of β.

(b) Let $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$.

i. [2 marks] Show that $\overline{Y} \sim \Gamma(n, \beta/n)$. (State without proof any property of moment-generating functions that you may want to use to prove this result.)

ii. [1 mark] Show that $\hat{\beta} = \overline{Y}$ is an unbiased estimator of β.

(c) [2 marks]
Find the relative efficiency of $\hat{\beta}$ with respect to $\tilde{\beta}$.

(d) [2 marks]
Make a statement about the consistency of both $\tilde{\beta}$ and $\hat{\beta}$.

4. [14 marks total]

(a) Let $X_1, \ldots, X_n$ be a random sample from a $\Gamma(\alpha, \beta)$ distribution, $\alpha > 0$ and $\beta > 0$ both unknown.

i. [4 marks] Find the method of moments estimators of α and β, say $\tilde{\alpha}$ and $\tilde{\beta}$.

ii. [3 marks] Show that $\tilde{\alpha}$ and $\tilde{\beta}$ are consistent. State clearly any result you use in doing so.

(b) Suppose now that $X_1, \ldots, X_n$ is a random sample from a $\Gamma(\alpha, \beta)$ distribution, this time with $\alpha > 0$ known and $\beta \geq 1$ unknown (**note the constraint on** β).

i. [5 marks] Find the maximum likelihood estimator of β, say $\hat{\beta}$.

ii. [2 marks] Find the maximum likelihood estimator of $\lambda = 1/\beta$.

5. [12 marks total]

The following data consist of height measurements in centimeters of 5 randomly selected male children at ages 3 and 5.

Child	Height at 3 years (x_i)	Height at 5 years (y_i)	Difference ($y_i - x_i$)
1	88.2	104.4	16.2
2	86.6	99.7	13.1
3	86.4	100.3	13.9
4	90.7	106.9	16.2
5	67.1	83.2	16.1
Observed sample mean	83.8	98.9	15.1
Observed sample variance	90.1	85.8	2.22

Assume that the data are observed from Normal populations.

(a) [5 marks]

Find a 95% confidence interval of the form $(a, +\infty)$ for the variance of the height of male children at 3 years.

(b) [7 marks]

Find a 95% confidence interval for the difference in height amongst boys between ages 3 and 5.

6. [12 marks total]

Let X and Y have the same joint probability density function as in Question **1.** We wish to test $H_0 : \theta = 1$ against $H_1 : \theta > 1$. We have two competing tests:

- Test 1: $X > 1 - \sqrt{0.05}$
- Test 2: $X > c$ and $Y > c$.

(a) [3 marks]

Show that the probability of a Type I error under Test 1 is 0.05.

(b) [4 marks]

Find c such that Test 2 has a probability of a Type I error of 0.05 as well.

(c) [5 marks]

Determine the power functions of both tests for $\theta > 1$. (If you have not found c in part **(b)**, express the power function of Test 2 in terms of c.)

CONTINUED

7. [8 marks total]

Consider the following observations of the number of metro trains going through a station between 8:00 and 8:15 over a period of 50 days:

Number of trains	Frequency
0	17
1	14
2	10
3	9
4 or more	0

Test the null hypothesis that these data come from a Poisson distribution at the $\alpha = 0.05$ level.

8. [12 marks total]

Consider the data from Question **5**. Let Y_i be the height of child i at 5 years, and x_i be the height of child i at age 3.

(a) [4 marks]

Use the observations to obtain least-squares estimates of α and β in the model

$$Y_i = \alpha + \beta x_i + \epsilon_i$$

for $\epsilon_i \sim N(0, \sigma^2)$, $\sigma^2 > 0$ unknown.

(b) [4 marks]

Estimate the variances of the estimators in part **(a)**.

(c) [4 marks]

Test $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$ at the 5% level.

9. [10 marks total; 1 mark per question]
Answer the following by true (T) or false (F). Do not justify your answer.

(a) If Y is a continuous random variable, then $\mathrm{P}[X \le x | Y = y] = 0$ for any random variable X.

(b) If $X_1, \ldots, X_n$ is a $N(\mu, \sigma^2)$ random sample with μ and σ^2 known, then $\sqrt{n}\dfrac{\overline{X} - \mu}{\sqrt{S^2}} \sim N(0, 1)$.

(c) Let X have probability density function $f(x) = \left(\frac{x}{\theta}\right)^{\alpha - 1} \mathbb{1}[0 < x < \theta]$. Then θX is a pivot for θ.

(d) A sequence of random variables X_n is consistent if and only if $\mathrm{Var}[X_n] \to 0$ and $\mathrm{Bias}(X_n) \to 0$ as $n \to \infty$.

(e) Let $\mathbf{X} = [X_1, \ldots, X_n]$ be a random sample from a distribution with parameter $\theta \in \Theta$. We wish to test H_0 against H_1, where $H_0 \cup H_1 = \Theta$. Then the likelihood ratio statistic of H_0 vs. H_1 is given by

$$\lambda(\mathbf{X}) = \frac{L(\hat{\theta}_0 | \mathbf{X})}{L(\hat{\theta} | \mathbf{X})},$$

where $L(\theta|\mathbf{X})$ is the likelihood of θ, $\hat{\theta}_0$ is the maximum likelihood estimator of θ for $\theta \in H_0$, and $\hat{\theta}$ is the maximum likelihood estimator of θ.

(f) We obtain a 95% confidence interval $[a, b]$ for a parameter θ, where a and b are observed values of some statistics. Then $\mathrm{P}[a \le \theta \le b] = 0.95$.

(g) We have a random sample $X_1, \ldots, X_n$ from an $\mathrm{Exp}(\beta)$ distribution (see page 6). We wish to test $H_0 : \beta = \beta_0$ vs. $H_1 : \beta = \beta_1$ with $0 < \beta_1 < \beta_0$. Then for any probability of Type I error, the test with the greatest power at β_1 will have a rejection region of the form $\overline{X} < c$ for some c.

(h) Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$, $\mu \in \mathbb{R}$ unknown and $\sigma^2 > 0$ known. Then the sample mean and sample variance of the random sample are independent.

(i) Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$, $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ both unknown. Then the sample mean and sample variance of the random sample are independent.

(j) In a simple linear regression setting, the method of least squares and the method of maximum likelihood yield identical estimates.

END OF FINAL EXAMINATION.
A list of pmf's and pdf's follows.

List of probability mass functions and probability density functions:

- Beta **Beta**

  - If $X \sim \text{Beta}(\alpha, \beta)$, then X has probability density function

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}[0 < x < 1]$$

    for $\alpha, \beta > 0$. See Gamma distribution for properties of the Gamma function.
  - $\mathbb{E}[X] = \dfrac{\alpha}{\alpha+\beta}$.
  - $\text{Var}[X] = \dfrac{\alpha\beta}{(\alpha+\beta+1)^2(\alpha+\beta+2)}$.

- Bin **Binomial**

  - If $X \sim \text{Bin}(n, p)$, then X has probability function

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}\mathbb{1}[x \in \{0, 1, \ldots, n\}],$$

    where $n = 1, 2, \ldots$ and $0 < p < 1$.
  - $\mathbb{E}[X] = np$.
  - $\text{Var}[X] = np(1-p)$.
  - MGF: $M_X(t) = [1 - p + p\exp(t)]^n$.

- χ^2_ν **Chi-squared**

  - If $X \sim \chi^2_\nu$, then X has probability density function given by

$$f(x) = \frac{x^{\nu/2-1}}{2^{\nu/2}\Gamma(\nu/2)}\exp(-x/2)\mathbb{1}[x > 0].$$

    with $\nu > 0$.
  - See Gamma distribution for properties, as $X \sim \chi^2_\nu \Leftrightarrow X \sim \Gamma\left(\frac{\nu}{2}, 2\right)$.

- Exp **Exponential**

  - If $X \sim \text{Exp}(\beta)$, then X has probability density function

$$f(x) = \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right)\mathbb{1}[x > 0]$$

    for $\beta > 0$.

- CDF: $F(x) = \begin{cases} 1 - \exp(-x/\beta) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$
- See Gamma distribution for other properties, as $X \sim \text{Exp}(\beta) \Leftrightarrow X \sim \Gamma(1, \beta)$.

-Γ **Gamma**

- If $X \sim \Gamma(\alpha, \beta)$, then X has probability density function

$$f(x) = \frac{1}{\Gamma(\alpha)} \frac{1}{\beta^\alpha} x^{\alpha-1} \exp(-x/\beta) \mathbb{1}[x > 0]$$

for $\alpha, \beta > 0$, where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp(-x) dx$. The Gamma function $\Gamma(\alpha)$ has the following properties:

  - $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ for $\alpha > 0$;
  - $\Gamma(1) = 1$;
  - If n is a positive integer, $\Gamma(n + 1) = n!$;
  - $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
- $\mathbb{E}[X] = \alpha\beta$.
- $\text{Var}[X] = \alpha\beta^2$.
- MGF: $M_X(t) = (1 - \beta t)^{-\alpha}$.

- Geom **Geometric**

- If $X \sim \text{Geom}(p)$, then X has probability mass function

$$f(x) = p(1 - p)^x$$

for $p \in [0, 1]$ and $x \in 0, 1, 2, \ldots$.

- $\mathbb{E}[X] = \dfrac{1 - p}{p}$.
- $\text{Var}[X] = \dfrac{1 - p}{p^2}$.
- MGF: $M_X(t) = \frac{p}{1-(1-p)e^t}$.

- N **Normal**

- If $X \sim \text{N}(\mu, \sigma^2)$, then X has probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(x - \mu)^2\right] \mathbb{1}[x \in \mathbb{R}]$$

for $\mu \in \mathbb{R}$ and $\sigma^2 > 0$.

- $\mathbb{E}[X] = \mu$.
- $\text{Var}[X] = \sigma^2$.
- MGF: $M_X(t) = \exp\left(\mu t + \sigma^2 \frac{t^2}{2}\right)$.

- NB **Negative Binomial**

- If $X \sim \text{NB}(r, p)$, then X has probability function

$$f(x) = \begin{cases} \binom{x-1}{r-1} p^r (1-p)^{x-r} & \text{if } x = r, r+1, r+2, \ldots \\ 0 & \text{otherwise,} \end{cases}$$

for $r = 1, 2, \ldots,$ and $0 < p < 1$.

- $\mathbb{E}[X] = \frac{r}{p}$.
- $\text{Var}[X] = \frac{r(1-p)}{p^2}$.
- MGF: $M_X(t) = \left(\frac{pe}{1-(1-p)e^t}\right)^r$.

- Po **Poisson**

- If $X \sim \text{Po}(\lambda)$, then X has probability function

$$f(x) = \exp(-\lambda)\frac{\lambda^x}{x!}\mathbb{1}[x \subset \{0, 1, 2, \ldots\}]$$

for $\lambda > 0$.

- $\mathbb{E}[X] = \lambda$.
- $\text{Var}[X] = \lambda$.
- MGF: $M_X(t) = \exp[\lambda(e^t - 1)]$

END OF LIST OF PMF's and PDF's

Tables of some sampling distributions follow.

Upper tail probabilities of the standard Normal distribution

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010

t distribution quantiles

Degrees of freedom	Upper tail probability 0.1	0.05	0.025	0.01	0.005
1	3.08	6.31	12.71	31.82	63.66
2	1.89	2.92	4.30	6.96	9.92
3	1.64	2.35	3.18	4.54	5.84
4	1.53	2.13	2.78	3.75	4.60
5	1.48	2.02	2.57	3.36	4.03
6	1.44	1.94	2.45	3.14	3.71
7	1.41	1.89	2.36	3.00	3.50
8	1.40	1.86	2.31	2.90	3.36
9	1.38	1.83	2.26	2.82	3.25
10	1.37	1.81	2.23	2.76	3.17
11	1.36	1.80	2.20	2.72	3.11
12	1.36	1.78	2.18	2.68	3.05
13	1.35	1.77	2.16	2.65	3.01
14	1.35	1.76	2.14	2.62	2.98
15	1.34	1.75	2.13	2.60	2.95
16	1.34	1.75	2.12	2.58	2.92
17	1.33	1.74	2.11	2.57	2.90
18	1.33	1.73	2.10	2.55	2.88
19	1.33	1.73	2.09	2.54	2.86
20	1.33	1.72	2.09	2.53	2.85
21	1.32	1.72	2.08	2.52	2.83
22	1.32	1.72	2.07	2.51	2.82
23	1.32	1.71	2.07	2.50	2.81
24	1.32	1.71	2.06	2.49	2.80
25	1.32	1.71	2.06	2.49	2.79
26	1.31	1.71	2.06	2.48	2.78
27	1.31	1.70	2.05	2.47	2.77
28	1.31	1.70	2.05	2.47	2.76
29	1.31	1.70	2.05	2.46	2.76
30	1.31	1.70	2.04	2.46	2.75
35	1.31	1.69	2.03	2.44	2.72
40	1.30	1.68	2.02	2.42	2.70
45	1.30	1.68	2.01	2.41	2.69
50	1.30	1.68	2.01	2.40	2.68
55	1.30	1.67	2.00	2.40	2.67
60	1.30	1.67	2.00	2.39	2.66
80	1.29	1.66	1.99	2.37	2.64
100	1.29	1.66	1.98	2.36	2.63
120	1.29	1.66	1.98	2.36	2.62
∞	1.28	1.64	1.96	2.33	2.58

χ^2 distribution quantiles

Degrees of freedom	Upper tail probability 0.1	0.05	0.025	0.01	0.005
1	2.71	3.84	5.02	6.63	7.88
2	4.61	5.99	7.38	9.21	10.60
3	6.25	7.81	9.35	11.34	12.84
4	7.78	9.49	11.14	13.28	14.86
5	9.24	11.07	12.83	15.09	16.75
6	10.64	12.59	14.45	16.81	18.55
7	12.02	14.07	16.01	18.48	20.28
8	13.36	15.51	17.53	20.09	21.95
9	14.68	16.92	19.02	21.67	23.59
10	15.99	18.31	20.48	23.21	25.19
11	17.28	19.68	21.92	24.72	26.76
12	18.55	21.03	23.34	26.22	28.30
13	19.81	22.36	24.74	27.69	29.82
14	21.06	23.68	26.12	29.14	31.32
15	22.31	25.00	27.49	30.58	32.80
16	23.54	26.30	28.85	32.00	34.27
17	24.77	27.59	30.19	33.41	35.72
18	25.99	28.87	31.53	34.81	37.16
19	27.20	30.14	32.85	36.19	38.58
20	28.41	31.41	34.17	37.57	40.00
21	29.62	32.67	35.48	38.93	41.40
22	30.81	33.92	36.78	40.29	42.80
23	32.01	35.17	38.08	41.64	44.18
24	33.20	36.42	39.36	42.98	45.56
25	34.38	37.65	40.65	44.31	46.93
26	35.56	38.89	41.92	45.64	48.29
27	36.74	40.11	43.19	46.96	49.64
28	37.92	41.34	44.46	48.28	50.99
29	39.09	42.56	45.72	49.59	52.34
30	40.26	43.77	46.98	50.89	53.67
35	46.06	49.80	53.20	57.34	60.27
40	51.81	55.76	59.34	63.69	66.77
45	57.51	61.66	65.41	69.96	73.17
50	63.17	67.50	71.42	76.15	79.49
55	68.80	73.31	77.38	82.29	85.75
60	74.40	79.08	83.30	88.38	91.95
80	96.58	101.88	106.63	112.33	116.32
100	118.50	124.34	129.56	135.81	140.17
120	140.23	146.57	152.21	158.95	163.65