McGill UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAM

MATH 323

PROBABILITY THEORY

Examiner: Professor X. Zhang
Associate Examiner: Professor G. Styan

Date: Tuesday December 21, 2004
Time: 2:00 p.m -5:00 p.m

INSTRUCTIONS

1. Please answer all 6 questions.
2. Please answer in the exam booklets provided.
3. This is a closed book exam.
4. No books or notes are permitted
5. No Dictionaries are allowed.
6. Standard calculators are allowed.
7. This exam consists of the cover page and 2 pages of 6 questions and a page of Normal tables has been provided..

1. Scores on an examination are assumed to be normally distributed with mean 78 and variance 36.

   (a) (5 marks) What is the probability that a person taking the examination scores higher than 72?

   (b) (5 marks) Suppose that students scoring the top 10% of this distribution are to receive an A grade, what is the minimum score a student must achieve to earn an A grade?

   (c) (5 marks) What must be the cutoff point for passing the examination if the examiner wants only the top 28.1% of all scores to be passing?

   (d) (5 marks) Approximately what proportion of students have scores 5 or more points above the score that cuts off the lowest 25%?

2. (10 marks) A bank uses a screening system for issuing home mortgages. The borrowers who repay their loans are characterized as "good" and those who default "bad". The screening system accurately identifies good borrowers with probability 98% and bad borrowers with probability 80%. Historical evidence suggests that, without screening, 76% of the borrowers will repay their loans. Suppose that a borrower passes the screening, what is the probability that the borrower will repay his or her loan?

3. Two fair coins are tossed at the same time. For i=1, 2, define $X_i = 1$ if the i^{th} coin lands a head, 0 if a tail. Let $X=X_1$ and $Y=X_1+X_2$.

   (a) (5 marks) Find the joint probability distribution of X and Y.

   (b) (5 marks) Find the marginal distributions of X and Y respectively.

   (c) (5 marks) Find $E(Y|X=0)$ and $E(Y|X=1)$.

4. Suppose that the weather can be characterized by either "rain" (R) or "shine" (S). The probability of R or S tomorrow depends only on the weather today. If today is R then the probability of R tomorrow is 0.7; if today is S then the probability of R tomorrow is 0.4. Let X_n be the weather on the n^{th} day.

   (a) (3 marks) Show that $\{X_n\}$ is a Markov chain and find the transition probability matrix.

   (b) (5 marks) If it is S today, what is the probability of R the day after tomorrow?

   (c) (5 marks) Find the stationary distribution.

   (d) (2 marks) What is the approximate probability of R on a given day in the future?

5. (a) (5 marks) Show that the moment generating function for a normal distribution with mean μ and variance σ^2 is $e^{\mu t+\sigma^2 t^2/2}$.

   (b) (5 marks) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a normal distribution with mean μ and variance σ^2 and let $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, the sample mean. Find the moment generating function of $\overline{Y}$ and then the distribution of $\overline{Y}$.

   (c) (5 marks) Let $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$, the sample variance. Show that $E(S^2) = \sigma^2$.

   (d) (5 marks) It is known that $(n-1)S^2/\sigma^2$ has a χ^2 distribution with (n-1) degrees of freedom, with the moment generating function $(1-2t)^{-(n-1)/2}$. Show that S^2 satisfies the Weak Law of Large Numbers, that is, S^2 converges in probability to σ^2 as the sample size $n \to \infty$.

6. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a Bernoulli distribution with success probability p and let $\overline{Y}$ be the sample mean.

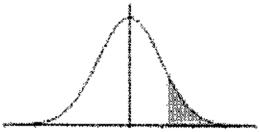
   (a) (5 marks) Show that, for any $\varepsilon>0$,

$$P(|\overline{Y} - p| \geq \varepsilon) \leq 1/(4n\varepsilon^2).$$

   (b) (5 marks) Suppose that it is desired that the sample mean $\overline{Y}$ be within 5% of the population mean p, with probability at least 90%. Using (a), find the sample size n.

   (c) (10 marks) Use the normal approximation to find the sample size n that is needed in (b).

Table A.3 – Areas in the upper tail of the standard normal distribution



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500	0.496	0.492	0.488	0.484	0.480	0.476	0.472	0.468	0.464
0.1	0.460	0.456	0.452	0.448	0.444	0.440	0.436	0.433	0.429	0.425
0.2	0.421	0.417	0.413	0.409	0.405	0.401	0.397	0.394	0.390	0.386
0.3	0.382	0.378	0.374	0.371	0.367	0.363	0.359	0.356	0.352	0.348
0.4	0.345	0.341	0.337	0.334	0.330	0.326	0.323	0.319	0.316	0.312
0.5	0.309	0.305	0.302	0.298	0.295	0.291	0.288	0.284	0.281	0.278
0.6	0.274	0.271	0.268	0.264	0.261	0.258	0.255	0.251	0.248	0.245
0.7	0.242	0.239	0.236	0.233	0.230	0.227	0.224	0.221	0.218	0.215
0.8	0.212	0.209	0.206	0.203	0.200	0.198	0.195	0.192	0.189	0.187
0.9	0.184	0.181	0.179	0.176	0.174	0.171	0.169	0.166	0.164	0.161
1.0	0.159	0.156	0.154	0.152	0.149	0.147	0.145	0.142	0.140	0.138
1.1	0.136	0.133	0.131	0.129	0.127	0.125	0.123	0.121	0.119	0.117
1.2	0.115	0.113	0.111	0.109	0.107	0.106	0.104	0.102	0.100	0.099
1.3	0.097	0.095	0.093	0.092	0.090	0.089	0.087	0.085	0.084	0.082
1.4	0.081	0.079	0.078	0.076	0.075	0.074	0.072	0.071	0.069	0.068
1.5	0.067	0.066	0.064	0.063	0.062	0.061	0.059	0.058	0.057	0.056
1.6	0.055	0.054	0.053	0.052	0.051	0.049	0.048	0.047	0.046	0.046
1.7	0.045	0.044	0.043	0.042	0.041	0.040	0.039	0.038	0.038	0.037
1.8	0.036	0.035	0.034	0.034	0.033	0.032	0.031	0.031	0.030	0.029
1.9	0.029	0.028	0.027	0.027	0.026	0.026	0.025	0.024	0.024	0.023
2.0	0.023	0.022	0.022	0.021	0.021	0.020	0.020	0.019	0.019	0.018
2.1	0.018	0.017	0.017	0.017	0.016	0.016	0.015	0.015	0.015	0.014
2.2	0.014	0.014	0.013	0.013	0.013	0.012	0.012	0.012	0.011	0.011
2.3	0.011	0.010	0.010	0.010	0.010	0.009	0.009	0.009	0.009	0.008
2.4	0.008	0.008	0.008	0.008	0.007	0.007	0.007	0.007	0.007	0.006
2.5	0.006	0.006	0.006	0.006	0.006	0.005	0.005	0.005	0.005	0.005
2.6	0.005	0.005	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004
2.7	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003
2.8	0.003	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002
2.9	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.001	0.001	0.001
3.0	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
3.1	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
3.2	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
3.3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3.4	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000