1. (20 marks) Give definitions of contraction mapping, iterated function system, Hausdorff metric, and explain, giving examples, how an iterated function system gives rise to a fractal.

2. (20 marks) Describe the Deterministic Fractal Drawing Algorithm and the Randomized Fractal Drawing Algorithm and explain how and why each one works.

3. (20 marks) Find the fixed points, determine their stability and sketch the phase space for each of the following:

   (a) The discrete dynamical system given by $f(x) = x^2 - x + 1/2$.

   (b) The continuous dynamical system $x^1 = x^2 - 3x + 2$.

4. (20 marks) For the system of differential equations

$$\begin{aligned} x' &= y^2 - x \\ y' &= x^2 - y \end{aligned}$$

   (a) find the fixed points and determine their stability.

   (b) Show that for $0 < x_0 = y_0 < 1$, there is a solution $(x(t), y(t))$ satisfying

$$\begin{aligned} x(t) &= y(t) \text{ for all } t \\ x(0) &= x_0 = y_0 = y(0) \end{aligned}$$

$$\begin{aligned} \lim_{t \to \infty} x(t) &= \lim_{t \to \infty} y(t) = 0 \\ \lim_{t \to -\infty} x(t) &= \lim_{t \to -\infty} y(t) = 1 \end{aligned}$$

   (c) Draw the phase space.

5. (20 marks) A fractal curve is generated as follows:

   Write down a corresponding IFS consisting of two maps, and compute the fractal dimension of the curve in two ways.

6. (20 marks)

   (a) Describe the construction of the Cantor Set $\mathcal{C}$ and list its properties.

   (b) Define chaos for a discrete dynamical system.

   (c) Show that the dynamical system

   $$f(x) = 3x \bmod 1 \qquad x \in \mathcal{C}$$

   is chaotic.

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-322A/382A

DYNAMICAL SYSTEMS, FRACTALS & CHAOS

Examiner: Professor R. Rigelhof

Date: Wednesday, December 15, 1999
Time: 9:00 A.M. - 12:00 Noon.

INSTRUCTIONS

Calculators are not permitted.
Students in 189-322A may do any FIVE questions.
Students in 189-382A must answer ALL questions.

This exam comprises the cover and two pages of questions.