McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 318

MATHEMATICAL LOGIC

Examiner: Professor M. Makkai
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Date: Monday December 17, 2007.
Time: 9:00 A.M - 12:00 P.M

INSTRUCTIONS

1. Please answer questions in the exam booklets provided.
2. No calculators are permitted.
3. This is a closed book exam.
4. Use of Regular and or translation dictionary are permitted.

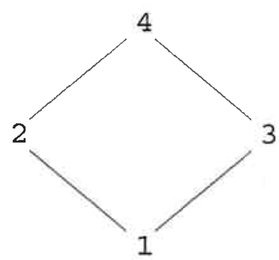
This exam consists of the cover page, 4 pages of questions and two pages of information that may be used.

[1](16.5%) Let (A, R) and (B, S) be binary relations. Define the binary relation (C, T) by $C=A\times B$, and

$$((a_1, b_1), (a_2, b_2)) \in T \overset{\text{DEF}}{\iff} (a_1, b_1)\in R \text{ and } (a_2, b_2)\in S.$$

Accept without proof that if (A, R) and (B, S) are (reflexive partial) orders, then so is (C, T).

(i) Let the orders (A, R) and (B, S) be given by the following Hasse diagrams:

(A, R) :


(B, S) :


Draw the Hasse diagram of the order (C, T), where (C, T) is constructed from (A, R) and (B, S) as described above.

(ii) **Prove** that if (A, R) and (B, S) are arbitrary lattices, and (C, T) is constructed from them as above, then, for every $x, y\in C$, the join $x\vee y$ exists in the order (C, T) [in fact, (C, T) is a lattice; but you don't need to give the complete proof of this fact.]

[2](16.5%) Give the definition of "Boolean algebra".

The definition should be complete. Start with a binary relation (A, R), and give all the conditions it needs to satisfy for it to be a Boolean algebra. You may use logical formulas to formulate some of the conditions, but it is not necessary to do so. All technical terms (e.g. "transitive", "complement", etc.) used in the definition have to be defined.

[3](16.5%) **(i)** Give a disjunctive form for the Boolean expression

$$(A \vee B) \wedge (C \vee D) \wedge (A \vee C) \wedge (B \vee D) \quad ;$$

make it as short as you can.

(ii) Decide if the Boolean entailment

$$(E \wedge B) \rightarrow D, \quad (C \wedge F) \rightarrow E, \quad A \rightarrow F, \quad (A \wedge D) \vee (A \wedge B \wedge C) \;\vdash\; A \wedge D$$

is correct. Make your calculations as economical as possible.

(iii) Calculate the full disjunctive normal form of the expression $\overline{AB \vee CD}$.

(iv) Suppose that for the subsets A, B, C and D of $\mathbb{R}^2$, we have that $(A \cap B) \cup (C \cap D) = \mathbb{R}^2$. $\langle A, B, C, D\rangle$ denotes the Boolean subalgebra of $(\mathcal{P}(\mathbb{R}^2), \subseteq)$ generated by the sets A, B, C and D. Use the result of (iii), and draw a conclusion on the number of elements of $\langle A, B, C, D\rangle$. Can you conclude that this number equals a certain numerical value; can you conclude that this number is at least or at most a certain value?

[4](16.5%) Let Φ be the sentence

$$\exists x(\exists yRxy \wedge \forall y(Rxy \longrightarrow \exists zRyz))\ .$$

Let

$U = \{1, 2, 3, 4\}$,
$R_1 = \{(1,2), (1, 4), (2, 3), (4, 3), (4, 4)\}$,
$R_2 = R_1 \cup \{(1, 3)\}$.

(i) **Evaluate** the truth-value of Φ in the interpretations (U, R_1) and (U, R_2) , by describing the computation the Tarski machine performs in both cases. In particular:

For each subformula Ψ of Φ (including Φ itself),

state what the set of free variables of Ψ is;

and

do one of (a) and (b), *whichever is more economical*:

(a) give the complete list of those tuples of values for the free variables of Ψ for which Ψ TRUE in (U, R_1) ;

(b) give the complete list of those tuples of values for the free variables of Ψ for which Ψ FALSE in (U, R_1) .

Notes: 1. For Φ itself, the set of variables is empty; there is a single value-tuple, the empty tuple; one gets a single truth-value.

2. Example: Rxy is a subformula of Φ . Its set of free variables is $\{x, y\}$. The value-tuples (x, y) for which Rxy is TRUE in (U, R_1) are $(1, 2)$, $(1, 4)$, $(2, 3)$, $(4, 3)$, $(4, 4)$. There are 11 value-tuples for which the truth-value of Rxy in (U, R_1) is FALSE ; it is more economical to do (a) in this case.

(ii) Repeat (i) for (U, R_2) in place of (U, R_1) . Take advantage of the fact that R_2 differs only in a small way from R_1 .

(iii) Give the NNF (negation normal form) and Skolem form for each of Φ and $\neg\Phi$, and specify Skolem functions for each case when (U, R_1) or (U, R_2) satisfies Φ or $\neg\Phi$.

[5](17%) Give a fully annotated deduction in the system of (extended) natural deduction (for the rules that may be used, see page 5 of this exam) for the following entailment:

$$\forall x \exists y (Pxy \vee Qxy),\ \forall x \forall y (Pxy \longrightarrow Rxy),\ \forall x \forall y (\neg(Qxy) \vee Rxy)\ \vdash\ \forall x \exists y Rxy\ .$$

[6](17%) **(i)** Show that the associative law for multiplication,

$$\forall x \forall y \forall z ((x \cdot y) \cdot z = x \cdot (y \cdot z))\ ,$$

is a theorem of Peano Arithmetic. In the proof, you may use the axioms, lemmas and theorems listed on page 6 of this exam, but nothing else. *There is no need to give a formal deduction.*

(ii) Replace P in the axiom schema by a particular formula, and write down, without abbreviations, the resulting instance of MI (see page 6) which is used in your proof.

Rules of Extended Natural Deduction

Abbreviations: $\Psi :=: \Psi_1, \ldots, \Psi_k$; $\Psi' :=: \Psi'_1, \ldots, \Psi'_\ell$. ($k=0$, $\ell=0$ possible)

Structural rules: **I:** $\dfrac{}{\Phi \vdash \Phi}$; **W:** $\dfrac{\Psi \vdash \Phi}{\Psi, \Gamma \vdash \Phi}$;

Int: $\dfrac{\Psi, \Gamma, \Lambda, \Psi' \vdash \Phi}{\Psi, \Lambda, \Gamma, \Psi' \vdash \Phi}$; **Con:** $\dfrac{\Psi, \Gamma, \Gamma \vdash \Phi}{\Psi, \Gamma \vdash \Phi}$; **Cut:** $\dfrac{\Psi \vdash \Lambda \qquad \Psi, \Lambda \vdash \Phi}{\Psi \vdash \Phi}$

T: $\dfrac{\Psi \vdash \Phi_1 \qquad \Psi \vdash \Phi_2 \qquad \ldots \qquad \Psi \vdash \Phi_k}{\Psi \vdash \Phi}$

Proviso: Φ is a tautological (Boolean) consequence of $\Phi_1, \ldots, \Phi_k$.

D: $\dfrac{\Psi, \Gamma \vdash \Phi}{\Psi \vdash \Gamma \rightarrow \Phi}$ **C:** $\dfrac{\Psi, \Phi \vdash \Gamma \qquad \Psi, \Phi \vdash \neg\Gamma}{\Psi \vdash \neg\Phi}$

AC: $\dfrac{\Psi, \Gamma \vdash \Phi \qquad \Psi, \Lambda \vdash \Phi}{\Psi, \Gamma \vee \Lambda \vdash \Phi}$

US: $\dfrac{\Psi \vdash \forall x \Phi}{\Psi \vdash \Phi[t/x]}$ **EG:** $\dfrac{\Psi \vdash \Phi[t/x]}{\Psi \vdash \exists x \Phi}$

Proviso for US and EG : the substitution $\Phi[t/x]$ is legal

UG: $\dfrac{\Psi \vdash \Phi}{\Psi \vdash \forall x \Phi}$ **Proviso:** the variable x is not free in Ψ.

ES: $\dfrac{\Psi, \Gamma \vdash \Phi}{\Psi, \exists x \Gamma \vdash \Phi}$ **Proviso:** the variable x is not free in Ψ, Φ.

CBV: $\dfrac{\Psi \vdash \Phi}{\Psi \vdash \Phi'}$: Φ' is obtained from Φ by a legal change of bound variables

E: $\dfrac{}{\emptyset \vdash t=t}$; $\dfrac{\Psi \vdash t_1=t_2 \qquad \Psi \vdash \Phi[t_1/x]}{\Psi \vdash \Phi[t_2/x]}$

$$\dfrac{\Psi \vdash t_1=t_2 \qquad \Psi \vdash \Phi[t_2/x]}{\Psi \vdash \Phi[t_1/x]}$$

A list of axioms, lemmas and theorems of Peano Arithmetic

*Ax*1. $x+0=x$

*Ax*2. $x+Sy=S(x+y)$

*Ax*3. $x\cdot 0=0$

*Ax*4. $x\cdot Sy=x\cdot y+x$

*AxSc*5. (*MI*; an *Axiom Schema*)

$$(P(0)\wedge\forall x(P(x)\longrightarrow P(Sx)))\longrightarrow\forall xP(x)$$

*Ax*6. $Sx\neq 0$

*Ax*7. $Sx=Sy\longrightarrow x=y$

*Thm*1. $(x+y)+z=x+(y+z)$

*L*1. $0+x=x$

*L*2. $S(x+y)=Sx+y$

*Thm*2. $x+y=y+x$

*Thm*3. $x\cdot(y+z)=(x\cdot y)+(x\cdot z)$