Student Name:
Student Id#:

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 317

NUMERICAL ANALYSIS

Examiner: Professor A.R. Humphries — Date: Monday December 14, 2009
Associate Examiner: Professor Gantumur Tsogtgerel — Time: 2:00 PM - 5:00 PM

INSTRUCTIONS

1. All questions carry equal weight.
2. Answer 6 or 7 questions; credit will be given for the best 6 answers.
3. Answer questions in the exam book provided. Start each answer on a new page.
4. This is a closed book exam.
5. Notes and textbooks are not permitted.
6. Non-programmable calculators are permitted.
7. Translation dictionaries (English-French) are permitted.

This exam comprises of the cover page, and 3 pages of 7 questions.

1. (a) Show that $f(x) = x^3 - 3$ has exactly one zero in the interval $[1, 2]$.

   (b) Starting from this interval with $x_0 = 1.5$, use three steps of the bisection method to obtain an approximation x_3 to this zero.

   (c) How many steps would be required to ensure that

   $$|x_n - x^*| \leqslant 10^{-8},$$

   where x^* is the zero of $f(x)$.

   (d) The following three iterative methods are proposed to compute $\sqrt[3]{3}$. Rank them in order, based on the order of convergence in a neighbourhood of the (positive) root.

   $$\text{(i)} \quad x_{n+1} = x_n - x_n^3 + 3, \quad \text{(ii)} \quad x_{n+1} = \frac{2x_n}{3} + \frac{1}{x_n^2}, \quad \text{(iii)} \quad x_{n+1} = x_n - \frac{(x_n^6 - 9)}{12x_n^2}.$$

2. (a) Let $f(x)$ be $n+1$ times continuously differentiable on $[a, b]$ and $x_0, x_1, \ldots, x_n$ be distinct interpolation points in $[a, b]$. Define the fundamental Lagrange polynomials $l_0(x), l_1(x), \ldots, l_n(x)$ for the interpolation points and show that

   $$p_n(x) = \sum_{j=0}^{n} f(x_j) l_j(x)$$

   interpolates f at $x_0, x_1, \ldots, x_n$.

   (b) Show that $p_n(x)$ is the unique interpolating polynomial of degree n.

   (c) Suppose that $n = 3$

   $$x_0 = 0, \quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 4,$$
   $$f(x_0) = 1, \quad f(x_1) = \frac{1}{\sqrt{2}}, \quad f(x_2) = 0, \quad f(x_3) = -1.$$

   Find $p_3(x)$ and evaluate $p_3(3)$.

   (d) Find a bound for the error in this approximation of $f(3)$, when $\max_{x \in [0,3]} |f^{(4)}(x)| \leqslant 1/6$, using the error formula

   $$f(x) = p_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i).$$

3. (a) What is the key difference between Lagrange and Hermite interpolants? What is the difference between a clamped and a natural cubic spline?

   (b) A natural cubic spline S on $[1, 3]$ has the formula

   $$S(x) = \begin{cases} S_0(x) = 4 + (x-1) - (x-1)^3, & \text{if } 1 \leqslant x \leqslant 2 \\ S_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^3, & \text{if } 2 \leqslant x \leqslant 3. \end{cases}$$

   Find a, b, c, d.

   (c) A cubic Bezier curve $\mathbf{B}(t)$ has end points $\mathbf{b}_0 = (0, 0)$ and $\mathbf{b}_3 = (1, 0)$ and guide points $\mathbf{b}_1 = (0, 1/2)$ and $\mathbf{b}_2 = (1, 1/2)$. What is the role of the guide points and what properties does the curve have with respect to the four given vectors? State the formula of the curve $\mathbf{B}(t)$.

4. (a) Show that

$$f''(x_0) = \frac{1}{h^2}\Big(f(x_0+h) - 2f(x_0) + f(x_0-h)\Big) - \frac{h^2}{12}f^{(4)}(\xi),$$

where $\xi \in [x_0 - h, x_0 + h]$.

(b) Given the data

x	0.9	1	1.1
$f(x)$	-0.0948	0	0.1048

Use the finite difference formula above to approximate $f''(1)$.

(c) Suppose $|f^{(4)}(x)| \leqslant M$ for all x, and that $h > 0$. If we encounter roundoff errors δ_i in computing $f(x_0 + ih)$ for $i = -1, 0, 1$ and and $|\delta_i| < \delta$, find an upper bound on the total error in the approximation to $f''(x_0)$. Determine the value of h which minimises this bound, if $\delta = 1 \times 10^{-16}$ and $M = 3$.

5. (a) Define the degree of accuracy (also known as the degree of precision) of a quadrature formula $I_h(f)$ for approximating the integral

$$I(f) = \int_a^b f(x)dx.$$

(b) Find the degree of accuracy p of the quadrature formula

$$I_h(f) = \frac{3}{8}h[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

where $a = x_0$, $b = x_3$ and $h = x_{i+1} - x_i$.

(c) Given that $I(f) = I_h(f) + kh^{p+2}f^{(p+1)}(\xi)$, where p is the degree of accuracy, find k.

(d) Evaluate $I_h(f)$ when $I(f) = \int_0^1 e^{-x^2}dx$.

6. (a) Let $I_h(f)$ be the Composite Trapezoidal Rule approximation to

$$\ln 2 = I(f) = \int_1^2 \frac{1}{x}dx.$$

Evaluate $I_h(f)$ when $h = 0.5$ and when $h = 0.25$.

(b) Derive the error bound

$$I(f) - I_h(f) = -\frac{(b-a)}{12}h^2 f''(\xi)$$

for some $\xi \in [a, b]$, for the Composite Trapezoidal rule, from the error bound for the Trapezoidal rule.

(c) Apply one-step of Richardson extrapolation to the approximations in (a), to find a better approximation to $\ln 2$.

(d) Given that $\ln 2 = 0.69314718$, compute the relative error in the approximations found in (a) and (c).

7. Consider the initial value problem

$$y' = f(y), \qquad 0 \leqslant t \leqslant T, \quad y(0) = \alpha.$$

Suppose you approximate the solution $y(t)$ using the Runge-Kutta method

$$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i + \frac{h}{2}\Big(f(w_i) + f(w_i + hf(w_i))\Big), \quad i = 0, ...N \end{aligned}$$

with time-step $h > 0$.

(a) Define the local truncation error $\tau_{i+1}(h)$ and use it to determine the order of this method.

(b) Consider the case where

$$f(y) = \lambda y, \quad \lambda < 0,$$

and

i. show that $w_{i+1} = (1 + h\lambda + \frac{(h\lambda)^2}{2})w_i$.

ii. Under what conditions on h does $\lim_{i\to\infty} w_i = 0$?