FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS MATH 315

Ordinary Differential Equations

Examiner: Professor S. W. Drury — Date: Tuesday, 28 April 2009

Associate Examiner: Professor N. Sancho — Time: 2: 00 pm. – 5: 00 pm.

INSTRUCTIONS

Answer all questions in the booklets provided.
You are expected to simplify your answers wherever possible.
This is a closed book examination.
Faculty standard calculators are permitted.
Both regular and translation dictionaries are permitted.

This exam has 7 questions on pages 2 and 3.
A Laplace Transform Table can be found on page 4.

1. (i) (8 points) Solve the initial value problem

$$x\frac{dy}{dx} = 3y + x^8, \qquad y(1) = 1.$$

(ii) (12 points) Find the general solution to the equation

$$2\Big(y\cos x + (\cos x)^2\Big)\frac{dy}{dx} + y^2 \sin x = 0.$$

Show that there is no solution which satisfies the initial condition $y(0) = -1$.

2. Find the general solution of the equations

(i) (10 points) $y\dfrac{d^2y}{dx^2} = 2\left(\dfrac{dy}{dx}\right)^2$.

(ii) (10 points) $x^2\dfrac{d^2y}{dx^2} + 7x\dfrac{dy}{dx} + 9y = 0$.

3. (20 points) Given that $y_1(x) = x$ and $y_2(x) = x^3 + 1$ are solutions of the equation

$$(2x^3 - 1)\frac{d^2y}{dx^2} - 6x^2\frac{dy}{dx} + 6xy = 0,$$

find the general solution of the equation

$$(2x^3 - 1)\frac{d^2y}{dx^2} - 6x^2\frac{dy}{dx} + 6xy = x(2x^3 - 1)^2.$$

4. Use the *Method of Undetermined Coefficients* to find the solution of the initial value problem

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 4xe^{2x}, \qquad y(0) = 0,\ \frac{dy}{dx}(0) = 1.$$

5. (i) (14 points) Find a fundamental set of solutions of the equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 8x\frac{dy}{dx} - 6y = 0$$

expressed as series in powers of x.

(ii) (6 points) Find the radii of convergence of the solutions you have obtained.

6. Consider the equation

$$x^2\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 6y = 0.$$

(i) (4 points) Explain why $x = 0$ is a regular singular point for this equation.

(ii) (6 points) Find the general recurrence relation and the indicial equation.

(iii) (10 points) Using the *smaller* indicial root, find a solution that is unbounded near $x = 0$.

Note: The series expansion should terminate before you reach the *bad spot*.

7. Suppose that a function g is defined by

$$g(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \le t \le 2 \\ 2 & \text{if } t > 2 \end{cases}$$

(i) (5 points) Find the Laplace transform $\mathcal{L}g$ of g.

(ii) (10 points) Find the Laplace transform $\mathcal{L}y$ of the solution y of the initial value problem

$$y'' + 3y' + 2y = g(t), \qquad y(0) = 0, y'(0) = -1.$$

(iii) (5 points) Find the solution y itself.

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Elementary Laplace Transforms

$f(t)$	$\mathcal{L}f(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
t^n	$\frac{n!}{s^{n+1}}, \quad s > 0 \ (n = 0, 1, \ldots)$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
$u_c(t)f(t-c)$	$e^{-cs}\mathcal{L}f(s), \quad s > 0$
$\delta_c(t)$	$e^{-cs}, \quad s > 0$
$e^{ct}f(t)$	$\mathcal{L}f(s-c)$
$f'(t)$	$s\mathcal{L}f(s) - f(0)$
$f''(t)$	$s^2\mathcal{L}f(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^n\mathcal{L}f(s) - s^{n-1}f(0) - \cdots f^{(n-1)}(0)$