McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 315

ORDINARY DIFFERENTIAL EQUATIONS

Examiner: Professor J.J Xu
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Date: Friday April 13, 2007
Time: 2:00PM - 5:00PM

INSTRUCTIONS

This is a closed book exam
Answer all questions in the exam booklets provided.
Faculty standard calculators are permitted.
A table of Laplace Transforms has been provided.

This exam comprises the cover, 3 pages of 8 questions and 1 page of tables, PRINTED Double-Sided.

1. **(10pts)** Given the equation

$$2y(y+2x^2)\mathrm{d}x + x(4y+3x^2)\mathrm{d}y = 0.$$

   (a) Show that this is not an exact equation,

   (b) Determine the values of the constants α and β, such that $\mu(x,y) = x^\alpha y^\beta$ is an integrating factor for this equation;

   (c) By using the integral factor found above, derive the general solution of the equation.

2. **(5pts)** Perform the phase line analysis for the the following autonomous equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(y-1)^2(y-3),$$

   and determine that

   - its equilibrium states;
   - the type of each equilibrium state,
   - the stability property of each equilibrium state,
   - sketch the integral curves in the physical plane (t, y), based on the above phase line analysis without solving the equation.

3. **(15pts)** Find the general solution for the following equations:

   (a) $(D^2 - 2D + 2)^2(D^2 - 1)y = 0;$

   (b) $(D^2 + 4)y = 16x\cos 2x;$

4. **(15pts)**

   (a) Find all values of α for which all solutions of

$$x^2y'' + \alpha xy' + \frac{5}{2}y = 0$$

   approach to zero as $x \to \infty$.

(b) Find the general solution for the following equations by using the method of variation of parameters:

$$x^2y'' - 4xy' + 6y = x^4 \sin x, \quad (x > 0).$$

5. **(15pts)**

Given the following equation

$$x^2y'' + \frac{1}{2}(x + \sin x)y' + y = 0,$$

(a) Find all the regular singular points

(b) Derive the indicial equation and the exponents at the singularity for each regular singular point;

(c) Determine whether the given equation has a solution that is bounded near the regular singular point, has all solutions bounded near the regular singular point, or has no non-zero solution bounded near the regular singular point.

6. **(10pts)** Given the equation

$$2xy'' + y' + xy = 0,$$

(a) Show that $x = 0$ is a regular singular point of the given equation and give the roots of the indicial equation;

(b) Determine the recurrence formula for the coefficients in the Frobenius series expansion of the solution near $x = 0$;

(c) Find at least the first four terms of two linear independent solutions: $y_1(x), y_2(x)$.

7. **(10pts)** (Choose one from two problems. You may get bonus points, if you solved two.) Find the Laplace transform of the following functions:

   (a)
   $$f(t) = 4\cos^2 bt, \quad (b \text{ constant});$$

   (b)
   $$f(t) = \begin{cases} 0, & 0 \le t \le 1 \\ t, & 1 < t \le 2 \\ 0, & t > 2. \end{cases}$$

8. **(10pts)** (Choose one from two problems. You may get bonus points, if you solved two.) Find the inverse Laplace transform of the following functions:

   (a)
   $$F(s) = \frac{2s+3}{(s-2)(s^2+1)},$$

   (b)
   $$F(s) = \frac{e^2 e^{-4s}}{2s-1}.$$

9. **(10pts)** Solve the following IVP's with the Laplace transform method:

$$y'' + 4y = \sin t - u_{2\pi}(t)\sin(t-2\pi), \; y(0) = 0, \; y'(0) = 0.$$

LAPLACE TRANSFORM TABLE

$f(t)$	$\mathcal{L}f(s)=\int_0^\infty f(t)e^{-st}dt$	$f(t)$	$\mathcal{L}f(s)=\int_0^\infty f(t)e^{-st}dt$
$t^n,\ n\in\mathbb{N}$	$\frac{n!}{s^{n+1}},\ \Re s>0$	$u_a(t),\ a>0$	$\frac{e^{-as}}{s},\ \Re s>0$
1	$s^{-1},\ \Re s>0$	$u_a(t)g(t-a),\ a>0$	$e^{-as}\mathcal{L}g(s)$
t	$s^{-2},\ \Re s>0$	$e^{at}g(t)$	$\mathcal{L}g(s-a)$
$t^\nu,\ \nu>-1$	$\frac{\Gamma(\nu+1)}{s^{\nu+1}},\ \Re s>0$	$g(ct),\ c>0$	$\frac{1}{c}\mathcal{L}g\left(\frac{s}{c}\right)$
e^{at}	$(s-a)^{-1},\ \Re s>a$	$\delta_a(t)=\delta(t-a),\ a>0$	$e^{-as},\ \Re s>0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2},\ \Re s>0$	$f*g(t)=\int_0^t f(t-u)g(u)du$	$\mathcal{L}f(s)\mathcal{L}g(s)$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2},\ \Re s>0$	$f^{(n)}(t),\ n\in\mathbb{N}$	$s^n\mathcal{L}f(s)-\sum_{k=0}^{n-1}s^{n-k-1}f^{(k)}(0)$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2},\ \Re s>a$	$\frac{df}{dt}(t)$	$s\mathcal{L}f(s)-f(0)$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2},\ \Re s>a$	$\frac{d^2f}{dt^2}(t)$	$s^2\mathcal{L}f(s)-sf(0)-f'(0)$
$t^ne^{at},\ n\in\mathbb{N}$	$\frac{n!}{(s-a)^{n+1}},\ \Re s>a$	$t^nf(t),\ n\in\mathbb{N}$	$(-1)^n\frac{d^n\mathcal{L}f}{ds^n}(s)$