McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 315

ORDINARY DIFFERENTIAL EQUATIONS

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Date: April 28, 2005
Time: 2:00PM - 5:00PM

INSTRUCTIONS

Answer all questions.
Faculty standard calculators are permitted.
A table of Laplace Transforms has been provided.

This exam comprises the cover, 2 pages of 6 questions and 1 page of tables.

1. **(15pts)** Solve the following IVP's:

   (i) $ty' + 2y = t^2 - t + 1, \quad (t > 0), \qquad y(1) = 1/2;$

   (ii) $y' = \frac{2y-3x}{2x+4y}, \qquad y(1) = 0.$

   (iii) Find the value of b for which the given equation is exact equation, and then solve it using that value of b.

$$(xy^2 + bx^2y)\mathrm{d}x + (x+y)x^2\mathrm{d}y = 0, \quad y(1) = 1.$$

2. **(15pts)** Find the general solution for the following equations:

   (a) $y''' - 2y'' + y' = t^3 + 2e^t;$

   (b) $x^2y'' - 3xy' + 4y = x^2 \ln x, \quad (x > 0)$, (Use the method of variation of parameters).

3. **(15pts)** Given the following equation

$$x(x+3)^2y'' - 2(x+3)y' - xy = 0,$$

   (a) Classify each singular point of the given equation as regular or irregular;

   (b) Derive the indicial equation for each regular singular point;

   (c) For each regular singular point, determine which of the following occurs:

   (i) all solutions are bounded near the point;

   (ii) there is a non-zero solution, at least one, but not every, non-zero solution is bounded near the point;

   (iii) there is no non-zero solution bounded near the point.

4. **(20pts)** Given the equation

$$xy'' + y' - y = 0,$$

(a) Show that $x = 0$ is a regular singular point of the given equation;

(b) Determine the recurrence formula for the coefficients in the Frobenius series expansion of the solution near $x = 0$;

(c) Find at least the first four terms of two linear independent solutions: y_1, y_2.

5. **(15pts)**

(a) Find the Laplace transform of the following functions:

$$\text{(i)} \quad f(t) = te^t \sin 5t, \qquad \text{(ii)} \quad f(t) = t\sin(bt);$$

(b) Find the inverse Laplace transform of the following functions:

$$\text{(i)} \quad F(s) = \frac{2s+1}{s^2 - 2s + 2}, \qquad \text{(ii)} \quad F(s) = \frac{2e^{-2s}}{s^2 - 4}.$$

6. **(20pts)** Solve the following IVP's with the Laplace transform method:

(a) $y'' - 2y' + 2y = \cos t, \; y(0) = 1, \; y'(0) = 0;$

(b) $y'' + y' + \frac{5}{4}y = g(t), \; y(0) = 0, \; y'(0) = 0$, where

$$g(t) = \begin{cases} \sin t & 0 \le t < \pi, \\ 0 & \pi < t. \end{cases}$$

Laplace Transform Table

$f(t)$	$\mathcal{L}f(s)=\int_0^\infty f(t)e^{-st}dt$	$f(t)$	$\mathcal{L}f(s)=\int_0^\infty f(t)e^{-st}dt$
$t^n,\ n\in\mathbb{N}$	$\frac{n!}{s^{n+1}},\ \Re s>0$	$u_a(t),\ a>0$	$\frac{e^{-as}}{s},\ \Re s>0$
1	$s^{-1},\ \Re s>0$	$u_a(t)g(t-a),\ a>0$	$e^{-as}\mathcal{L}g(s)$
t	$s^{-2},\ \Re s>0$	$e^{at}g(t)$	$\mathcal{L}g(s-a)$
$t^\nu,\ \nu>-1$	$\frac{\Gamma(\nu+1)}{s^{\nu+1}},\ \Re s>0$	$g(ct),\ c>0$	$\frac{1}{c}\mathcal{L}g\left(\frac{s}{c}\right)$
e^{at}	$(s-a)^{-1},\ \Re s>a$	$\delta_a(t)=\delta(t-a),\ a>0$	$e^{-as},\ \Re s>0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2},\ \Re s>0$	$f*g(t)=\int_0^t f(t-u)g(u)du$	$\mathcal{L}f(s)\mathcal{L}g(s)$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2},\ \Re s>0$	$f^{(n)}(t),\ n\in\mathbb{N}$	$s^n\mathcal{L}f(s)-\sum_{k=0}^{n-1}s^{n-k-1}f^{(k)}(0)$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2},\ \Re s>a$	$\frac{df}{dt}(t)$	$s\mathcal{L}f(s)-f(0)$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2},\ \Re s>a$	$\frac{d^2f}{dt^2}(t)$	$s^2\mathcal{L}f(s)-sf(0)-f'(0)$
$t^ne^{at},\ n\in\mathbb{N}$	$\frac{n!}{(s-a)^{n+1}},\ \Re s>a$	$t^nf(t),\ n\in\mathbb{N}$	$(-1)^n\frac{d^n\mathcal{L}f}{ds^n}(s)$