Final Examination December 18, 1996 Mathematics 189-314A

1. (a) The function $F : {}^3 \rightarrow {}^2$ has the coordinate functions $f_1(x,y,z) = x^2 + e^y,\ f_2(x,y,z) = x + y\sin z$. Find the derivative $DF(\vec{a})$, where $\vec{a} = (1,1,\pi)$.

   (b) For a certain function $z = f(x,y)$ it is known that $f(1,2) = 3$, $\dfrac{\partial f}{\partial x}(1,2) = 2,\ \dfrac{\partial f}{\partial y}(1,2) = 5$. Make a reasonable estimate of $f(1.1, 1.8)$.

2. Let f, g, h be continuously differentiable functions. State a condition guaranteeing that the equations

$$x = f(u,v,w),\ y = g(u,v,w),\ z = h(u,v,w)$$

can be solved locally for u, v, w as differentiable functions of x, y, z, and show that

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} \dfrac{\partial g}{\partial v} & \dfrac{\partial g}{\partial w} \\ \dfrac{\partial h}{\partial v} & \dfrac{\partial h}{\partial w} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} & \dfrac{\partial f}{\partial w} \\ \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} & \dfrac{\partial g}{\partial w} \\ \dfrac{\partial h}{\partial u} & \dfrac{\partial h}{\partial v} & \dfrac{\partial h}{\partial w} \end{vmatrix}}.$$

3. Find the minimum of $f(x,y,z) = x^2 + y^2 + z^2$ subject to the condition that $ax + by + cz = d$.

4. (a) Show that if C is a closed curve, and $\vec{R}$ is the position vector of a variable point on C, $\displaystyle\oint_C \vec{R} \cdot d\vec{R} = 0$.

   (b) A force field $\vec{F} = \dfrac{x\vec{\imath} + y\vec{\jmath} + z\vec{k}}{(x^2+y^2+z^2)^{3/2}}$, $(x^2 + y^2 + z^2 \neq 0)$ acts on a particle P. Find the work done by the field in moving the particle from the point $(1,2,3)$ to the point $(2,3,5)$.

5. (a) Evaluate $\displaystyle\iint_D \frac{x}{4x^2+y^2}\, dA_{xy}$ where D is the region in the first quadrant bounded by the coordinate axes, the ellipse $4x^2 + y^2 = 16$ and the ellipse $4x^2 + y^2 = 1$.

   (b) A solid right circular cone has surface $x^2 + y^2 = z^2$ and altitude h. Find its mass if its density at a point is proportional to the distance from the vertex (origin).

6. (a) Evaluate the line integral $\int_C 3x dx + x dy$

$$x^2 + y^2 = 1$$

where C is the semi-circle $x^2 + y^2 = 1,\ y > 0$ and the diameter from $(-1, 0)$ to $(1, 0)$. (See fig.)

(b) State Green's Theorem and then verify it for the example in (a) by evaluating the double integral in the theorem.

7. Let S be a closed surface bounding a domain of D of volume 150 feet^3. If $\vec{R}$ is the radius vector from the origin to any point (x, y, z), and $\vec{n}$ denotes the unit normal directed outward from the domain, evaluate

$$\iint_S \vec{R} \cdot d\vec{S} \;=\; \iint_S \vec{R} \cdot \vec{n} dS.$$

8. Given the surface S determined by the parametric representation $x = uv,\ y = u + v,\ z = u^2 + v^2$, where (u, v) ranges over the interior and boundary of the triangle with vertices $(0, 0),\ (1, 0),\ (1, 1)$, and given the vector field $\vec{F} = x\vec{k}$, show by computing both sides that

$$\iint_S \text{curl } \vec{F} \cdot \vec{n} dS \;=\; \oint_C \vec{F} \cdot d\vec{R},$$

where $\vec{n}$ is a unit normal to S, and C is the boundary curve of S traversed in the appropriate direction.
What theorem asserts the truth of the above equality in general?

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-314A

ADVANCED CALCULUS

Examiner: Professor R. Vermes
Associate Examiner: Professor W.O.J. Moser

Date: Wednesday, December 18, 1996
Time: 2:00 P.M. - 5:00 P.M.

This exam comprises the cover and 2 pages of questions.