McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

Charles Roth

MATH 271

LINEAR ALGEBRA AND PARTIAL DIFFERENTIAL EQUATIONS

Examiner: Professor C. Roth
Associate Examiner: Professor T. Wihler

T. P. Wihler

Date: Friday December 14, 2007
Time: 9:00 AM- 12:00 PM

INSTRUCTIONS

1. Please answer all questions in the exam booklets provided.
2. This is a closed book examination. No books, crib sheets or lecture notes permitted.
3. Calculators are neither permitted nor required.
4. Use of a regular and/or translation dictionary is permitted.

This exam comprises the cover page, 2 pages of 6 questions and 1 page of useful information.

MATH 271 | Final Examination | Friday December 14, 2007
Professor C. Roth | | Time: 9:00 AM - 12:00 PM

1. (11 Marks)

   (a) Obtain carefully the nontrival null space of the operator $\mathcal{L} = \frac{d^4}{dx^4} - \alpha^4$, where α is a real positive constant.

   (b) Show that the solution of

   $$\frac{d^4y}{dx^4} = \alpha^4 y; y(0) = 0,\ y\prime(0) = 0,\ y(L) = 0,\ y\prime(L) = 0$$

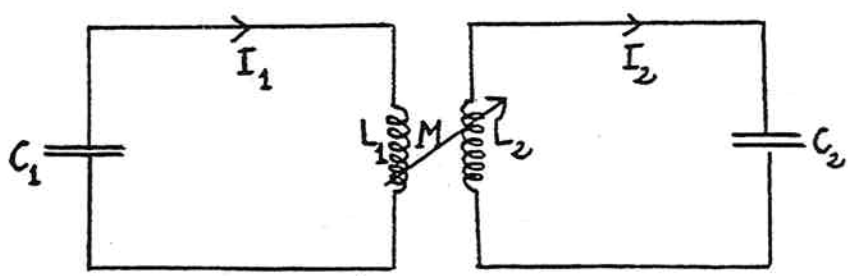
   is non-trivial provided $\cos \alpha L \cosh \alpha L = 1$.

2. (12 Marks)The circuit below is governed by the following system of differential equations

$$L_1\ddot{Q}_1 + M\ddot{Q}_2 + \frac{Q_1}{C_1} = 0$$

$$M\ddot{Q}_1 + L_2\ddot{Q}_2 + \frac{Q_2}{C_2} = 0.$$

If $L_1 = L_2 = 5$, $C_1 = C_2 = \frac{1}{8}$, $M = 3$, $Q_1(0) = 0$, $\dot{Q}_1(0) = 0$, $Q_2(0) = 4$, $\dot{Q}_2(0) = 0$, solve for $Q_1(t)$ and $Q_2(t)$.



3. (18 Marks)

   (a) Obtain the general solution of Laplace's equation in polar coordinates with the requirement that the solution be periodic of period 2π, i.e., solve

   $$\nabla^2\psi(r,\theta) = 0; \quad \psi(r,\theta + 2\pi) = \psi(r,\theta), \quad \psi_\theta(r,\theta + 2\pi) = \psi_\theta(r,\theta).$$

   (b) A very long cylinder of radius a is immersed in a uniform flow parallel to the x-axis with speed V_0. By having the center of the cylinder at the origin find the effect of the cylinder on the velocity potential and velocity. Assume that the cylinder is perpendicular to the flow.

4. (22 Marks) Solve and interpret physically the following diffusion equation boundary value problems:

   (a) $\psi_t - \psi_{xx} = h(x,t);\ 0 < x < \pi, t > 0$

   (i)$\psi_x(0,t) = F(t)$ (ii)$\psi_x(\pi,t) = G(t)$ (iii)$\psi(x,0) = f(x)$.

   (b) $\dfrac{1}{\alpha^2}\dfrac{\partial\psi}{\partial t}(r,\varphi,\theta,t) = \nabla^2\psi(r,\varphi,\theta,t);\ 0 \le r < a,\ 0 \le \varphi \le \pi,\ 0 \le \theta < 2\pi,\ \ t > 0$

   (i)$\psi(a,\varphi,\theta,t) = 0$ (ii)$\psi(r,\varphi,\theta,0) = 0$

5. (23 Marks) Solve the following Poisson's equation boundary value problems and interpret physically:

   (a) $\nabla^2\psi(x,y) = -\sin 3y;\quad 0 < x < \pi,\quad 0 < y < \pi.$

   (i) $\psi(0,y) = 0$, (ii)$\psi(\pi,y) = 0$, (iii)$\psi(x,0) = 1$, (iv)$\psi(x,\pi) = 0$

   (b) $\nabla^2\psi(r,\theta) = -2;\quad 0 \le r < 4,\quad 0 \le \theta < 2\pi.$

   (i)$\psi(4,\theta) = 0.$

6. (15 Marks) You may assume that the general solution of Laplace's equation in spherical coordinates, i.e., $\nabla^2\psi(r,\varphi) = 0, 0 \le \varphi \le \pi, \psi$ finite at $\varphi = 0$ and $\varphi = \pi$, is given by $\psi(r,\varphi) = \displaystyle\sum_{n=0}^{\infty}\left[A_n r^n + \frac{B_n}{r^{(n+1)}}\right] P_n(\cos\varphi)$, where P_n are the Legendre polynomials of order n.

   Solve $\nabla^2\psi(r,\varphi) = 0,\quad 0 \le \varphi < \pi/2,\ 0 \le r < \alpha,$

   (i) $\psi(r,\pi/2) = 0$, (ii) $\psi(\alpha,\varphi) = f(\cos\varphi)$ And as a special case $f(\cos\varphi) = \cos^3\varphi$.

   Leave your answer in SIMPLEST terms. Interpret physically.

USEFUL INFORMATION

1. $\nabla^2\psi(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\psi}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}.$

2.$\nabla\vec{\psi}(r.\theta) = \frac{\partial\psi}{\partial r}\hat{u}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{u}_\theta.$

3. Laplacian in spherical coordinates
$$\nabla^2\psi(r,\varphi,\theta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\phi}\left(\sin\varphi\frac{\partial\psi}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2\psi}{\partial\theta^2}.$$

Note: φ is the angle that the position vector makes with the z-axis.

4. $\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2\delta nm}{2n+1}$, where $P_n(x)$ and $P_m(x)$ are Legendre polynomials.

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$, $P_3(x) = \frac{1}{2}(5x^3-3x)$.

Good Luck!