1. (16 marks) Solve the following two problems:

   (a) $\nabla^2\psi(x,y,t) = \dfrac{1}{\alpha^2}\dfrac{\partial\psi}{\partial t}(x,y,t)$

   (b) $\nabla^2\psi(x,y) = -q(x,y)$

   for $0 < x < \pi,\ 0 < y < \pi$, with

   (i) $\psi(x,0) = 0$, (ii) $\psi(x,\pi) = 0$, (iii) $\psi_x(0,y) = 0$, (iv) $\psi_x(\pi,y) = 0$

   in both cases.

   (c) Give physical interpretations in both cases.

2. (14 marks) Solve

$$\nabla^2\psi(r,z) = 0; \qquad 0 \le r < b,\ 0 < z < \pi$$

   (i) $\psi(r,0) = 0$, (ii) $\psi(r,\pi) = f(r)$, (iii) $\psi(b,z) = g(z)$.

3. (11 marks) Find the potential distribution in the region below:

   Hint: Solve $\nabla^2\psi(r,\theta) = 0,\ 1 < r < e,\ 0 < \theta < \pi$.

   (i) $\psi(r,0) = 0$, (ii) $\psi(1,\theta) = 0$, (iii) $\psi(e,\theta) = 0$, (iv) $\psi(r,\pi) = f(r)$.

   Please leave your answer in simplest form.

4. (15 marks) A solid sphere of radius b is cooling into a medium of temperature $\beta°$. There is a constant heat generation at the rate Q. The initial temperature is $f(r)$. Find the temperature at any point inside the sphere after time t.

   Please leave your answer in simplest form.

   Hint: Solve $\dfrac{1}{\alpha^2}\dfrac{\partial\psi}{\partial t} - \nabla^2\psi = \dfrac{Q}{K};\ 0 \le r < b,\ t > 0.$

   (i) $\left[\dfrac{\partial\psi}{\partial r}\right]_{r=b} = h[T_0 - \psi(b,t)]$ where h is a positive constant.

   (ii) $\psi(r,0) = f(r)$, (iii) $\psi(r,t)$ is finite for $r = 0$.

5. Consider a cube of sides "a" with the x, y, z axes coinciding with the three intersecting edges of the cube.

   (a) (4 marks) Find the moments and products of inertia.

   (b) (10 marks) Find the principal moments of inertia and the directions of the principal axes.

   (c) (2 marks) Find the angle, and indicate how to obtain the axis (without performing detailed calculation for the latter), from the $[\hat{\imath}, \hat{\jmath}, \hat{k}]$ basis to the principal basis of inertia.

6. (a) (11 marks) Solve the system of differential equations

$$\dot{x}_1 = \frac{1}{2}x_1 - \frac{1}{8}x_2; \quad x_1(0) = 2$$
$$\dot{x}_2 = 2x_1 - \frac{1}{2}x_2; \quad x_2(0) = 3$$

   by using the exponential matrix method.

   (b) Write down Green's matrix for the system and discuss briefly its significance.

7. (12 marks) Solve

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{4}{t^2} & \frac{4}{t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t \\ 4 \end{bmatrix}; \; t \geq 1$$

with $X(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Good Luck!

McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATHEMATICS 189-266B

LINEAR ALGEBRA & BOUNDARY VALUE PROBLEMS

Examiner: Professor C. Roth — Date: Monday, April 17, 2000
Associate Examiner: Professor N.G.F. Sancho — Time: 9:00 A.M. - 1:00 P.M.

INSTRUCTIONS

Faculty Standard Calculators are permitted.

This exam comprises the cover, 2 pages of questions and one page of useful information.