1. (11 marks) Solve the vibrating string problem:

$$\begin{aligned} \psi_{xx} &= \psi_{tt}; & 0 < x < 1,\ t > 0 \\ \psi(0,t) &= \sin t, & \psi(1,t) = 0 \\ \psi(x,0) &= 0 & \psi_t(x,0) = 1 - x \end{aligned}$$

Leave your answer in simplest form.

2. (a) (5 marks) Determine the steady-state temperature distribution inside a sphere of radius "$a$" if the boundary temperature is maintained at a temperature of $T_0(1 + 2\sin^2\varphi)$.

   Leave your answer in simplest form. You may assume that the general solution of Laplace's equation in spherical coordinates with no $\theta$ dependence, and converging for $0 \le \varphi \le \pi$, is given by

$$\psi(r,\varphi) = \sum_{n=0}^{\infty} \left[ A_n r^n + \frac{B_n}{r^{n+1}} \right] P_n(\cos\varphi).$$

   (b) (7 marks) Solve and interpret physically

$$\nabla^2\psi(r,\varphi) = 0, \qquad r > a,\ 0 \le \varphi \le \pi.$$

   (i) $\lim_{r\to\infty} [\psi(r,\varphi) - V_0 r\cos\varphi] = 0.$ (ii) $\left[\frac{\partial\psi}{\partial r}\right]_{r=a} = 0.$

3. (11 marks) A sphere of radius $b$ has its surface maintained at a temperature $\beta$. There is a constant heat generation at the rate $Q$. The initial temperature is $f(r)$. Find the temperature at any point inside the sphere after time $t$.

   Leave your answer in simplest form.

   Hints: (a) $\psi = \psi(r,t)$; (ii) $\frac{1}{\alpha^2}\frac{\partial\psi}{\partial t} - \nabla^2\psi = \frac{Q}{K}.$

4. (10 marks) Find the steady-state temperature distribution in the region below:

5. Solve and interpret physically:

   (a) (12 marks) $\nabla^2\psi(r,z) = 0; \qquad 0 < r < b,\ 0 < z < \pi.$
   (i) $\psi(r,0) = 0,$ (ii) $\psi(r,\pi) = f(r),$ (iii) $\psi(b,z) = g(z).$
   Hint: Divide the problem into two parts.

   (b) (9 marks) $\nabla^2\psi(r,z) = -F(r,z); \qquad 0 < r < b,\ 0 < z < \pi.$
   (i) $\psi(r,0) = 0,$ (ii) $\psi(r,\pi) = f(r),$ (iii) $\psi(b,z) = g(z).$

6. (8 marks)A linear operator $\mathcal{T}$ maps the vector $5\hat{\imath} - 3\hat{\jmath}$ onto $24\hat{\imath} + \hat{\jmath}$ and the vector $-3\hat{\imath} + 7\hat{\jmath}$ onto $-4\hat{\imath} + 15\hat{\jmath}$. Through what angle must the $[\hat{\imath}, \hat{\jmath}]$ basis vectors be rotated in order for the matrix representative of $\mathcal{T}$ to be diagonal? It is sufficient to give the cosine of the required angle.

7. (a) (8 marks) The moment of inertia matrix is given by

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

   when referred to the $xyz$ coordinate system. Find the principal moments of inertia and unit vectors along the three principal axes.

   (b) (3 marks) Use matrix methods to identify and sketch

$$3x_1^2 + 3x_2^2 + 4x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3 = 36.$$

   (c) (3 marks) Obtain the matrix of transformation from the $(x_1, x_2, x_3)$ axes to those axes with respect to which the equation has no cross terms making certain that this transformation is a rotation. Explain.

   (d) (4 marks) Does

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-[3x^2+3y^2+4z^2+4xy+2xz+2yz]}dxdydz$$

   converge? If so, evaluate the integral.

8. (a) (6 marks) Solve the system

$$\begin{aligned} \dot{x}_1 &= 5x_1 + 4x_2; \quad x_1(0) = 2 \\ \dot{x}_2 &= -x_1 + x_2; \quad x_2(0) = -3 \end{aligned}$$

   by using the exponential matrix method.

   (b) (6 marks) Write down the Green's matrix for the above system and use it to solve

$$\begin{aligned} \dot{x}_1 &= 5x_1 + 4x_2 + 2; \quad x_1(0) = 2 \\ \dot{x}_2 &= -x_1 + x_2 - 1; \quad x_2(0) = -3. \end{aligned}$$

McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATHEMATICS 189-266B

LINEAR ALGEBRA & BOUNDARY VALUE PROBLEMS

Examiner: Professor C. Roth — Date: Wednesday, April 28, 1999

Associate Examiner: Professor N.G.F. Sancho — Time: 9:00 A.M. - 1:00 P.M.

INSTRUCTIONS

**Faculty Standard Calculators are permitted.**

This exam comprises the cover, 2 pages of questions and 1 page of useful information.