McGill UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 263

ORDINARY DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

Examiner: Professor P. Bartello
Associate Examiner: Professor J. Xu

Date: Friday December 17, 2004
Time: 2:00 P.M – 5:00 P.M

INSTRUCTIONS

1. Please answer all 7 questions.
2. Calculators are not permitted.
3. Please answer in exam booklets provided.
4. This is a closed book exam.
5. Translation and Regular Dictionaries are permitted.
6. This exam consists of the cover page and 2 pages of 7 questions and 1 Page of a Table.

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1. (15 marks)

   (a) Find the solution $y(x)$ of the differential equation

   $$2(xy^2 + xy)y' = y,$$

   subject to the initial condition $y(1) = 2$.

   (b) Find an implicit expression for the the general solution of the differential equation

   $$(2xe^{x^2}y^2 + \frac{\sin y}{x} - 9)dx + (2e^{x^2}y + \cos y \ln x)dy = 0.$$

   (c) Find the solution of the differential equation

   $$\frac{dy}{dx} = \frac{3y^2 + 9x^2}{2yx}, \qquad (x > 0)$$

   subject to the initial condition $y(1) = 1$.

2. (10 marks) Use variation of parameters to solve

   $$x^2y'' + xy' - 4y = 2x \quad (x > 0).$$

   Note: marks will be awarded based on the method.

3. (15 marks) Consider the equation

   $$y^{(4)} + \frac{1}{4}y'' = g(x).$$

   (a) Set $g(x) = 0$.

      i. Convert this to a second order ODE and find its general solution.

      ii. Using the Wronskian, verify that the solutions obtained are linearly independent.

   (b) Set $g(x) = 2x + 4$. Find the solution to the original fourth order ODE using undetermined coefficients.

4. (15 marks) (10 marks) Use Laplace transforms to solve the initial value problem

   $$y' - 2y = \begin{cases} \mathrm{t} & 0 \leq t < 5, \\ 5 & t \geq 5 \end{cases},$$

   with initial condition $y(0) = y_o$.

5. (10 marks) Find the inverse Laplace transform of

(a)

$$\frac{1}{s^4(s^2+1)}$$

(b)

$$\frac{s}{(s+1)(s^2+4)}$$

6. (a) (5 marks) Solve the system of equations

$$x_1 - 2x_2 + 3x_3 = 7$$

$$-x_1 + x_2 - 2x_3 = -5$$

$$2x_1 - x_2 - x_3 = 4.$$

(b) (5 marks) Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & -1 \\ 4 & -2 \end{pmatrix}.$$

7. (10 marks) Find the general solution of the system of differential equations

$$x_1' = x_1 + 4x_2$$

$$x_2' = \frac{x_1}{2} + 2x_2.$$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\dfrac{1}{s}, \quad s > 0$
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$
3. $t^n; \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \ p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
9. $e^{at} \sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}, \ n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t) f(t-c)$	$e^{-cs} F(s)$
14. $e^{ct} f(t)$	$F(s-c)$
15. $f(ct)$	$\dfrac{1}{c} F\left(\dfrac{s}{c}\right), \quad c > 0$
16. $\int_0^t f(t-\tau) g(\tau)\, d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$