MATH255 - Honours Analysis 2

Summary of Results

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Complete notes

1 Point-Set Topology

Topology is about abstracting openness. It can typically suffice to consider open, closed sets in $\mathbb{R}$ for intuition, but is obviously not all-general.

Definition 1 (Metric Space)**.** A space X equipped with a function $d : X \times X \to [0, \infty)$ is called a metric space and d a metric or distance if

- $d(x, y) = d(y, x) \geqslant 0$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) + d(y, z) \geqslant d(x, z)$

for any $x, y, z \in X$.

Definition 2 (Normed Vector Space)**.** A function $||\cdot|| : X \to \mathbb{R}$ defined on a vector space X over $\mathbb{R}$ is a norm if

- $||x|| \geqslant 0$
- $||x|| = 0 \iff x = 0$
- $||c \cdot x|| = |c|\,||x||$
- $||x + y|| \leqslant ||x|| + ||y||$,

for any $x, y \in X, c \in \mathbb{R}$.

Remark 1. We can naturally extend this to arbitary fields, but seeing as this is a course in Real Analysis, we won't.

Proposition 1. *For a normed vector space* $(X, ||\cdot||)$*,* $d(x, y) := ||x - y||$ *is a metric on* X*. We call such a metric the one "induced" by the norm.*

Definition 3 (Topological Set)**.** A set X is a topological set if we have a collection τ of subsets of X, called open sets, such that

- $\varnothing \in \tau, X \in \tau$
- For $A_\alpha \in \tau$ for α in any I (potentially infinite), $\bigcup_{\alpha \in I} A_\alpha \in \tau$
- For $A_\alpha \in \tau$ for $\alpha \in J$ where *J finite*, then $\bigcap_{\alpha \in J} A_\alpha \in \tau$

ie, arbitrary unions of open sets are open, and finite intersections of open sets are open.

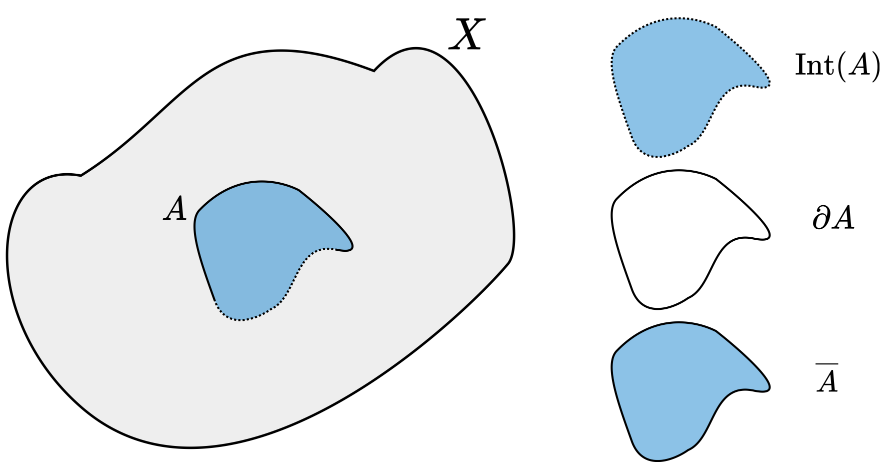
Remark 2. Keep $\mathbb{R}$ in mind when initially working with these definitions; for instance, the set $A_n := (0, \frac{1}{n})$ open in $\mathbb{R}$ for any $n \in \mathbb{N}$, but $\bigcap_{n \in \mathbb{N}} A_n = \{0\}$ which is closed.

Remark 3. Complemented each of these requirements gives similar definitions for closed sets of X.

Definition 4 (Topology on a Metric Space)**.** A subset $A \subseteq X$ open iff $\forall x \in A, \exists r = r(x) \in \mathbb{R}$, where $r(x) > 0$, such that $B(x, r(x)) := \{y \in x : d(x, y) < r(x)\} \subseteq A$. We call such a B an open ball, and $\overline{B}$ a closed ball with the same definition replacing the strict inequality with $\leqslant$.

Remark 4. While many of the spaces we look at our metric spaces that induce a topology as such, **not all topological spaces are metric spaces**. Indeed, "metrizability" (ie, equipping a topological space X with a metric that respects the open sets) is not a trivial activity.

Definition 5 (Equivalence of Metrics)**.** We say two metrics on X are equivalent if they admit the same topology; a sufficient condition is that, $\forall x \neq y \in X$, $\exists 1 < C < \infty$ such that $\frac{1}{C} < \frac{d_1(x,y)}{d_2(x,y)} < C$, then d_1, d_2 equivalent, where C independent of x, y.



Definition 6 ($\star$ Interior, Boundary, Closure)**.** Let X-topological space, $A \subseteq X$, $x \in X$.

- If $\exists U$-open s.t. $x \in U \subseteq A$, then we write $x \in \text{Int}(A)$, the interior of A.
- If $\exists V$-open s.t. $x \in V \subseteq A^C$, then $x \in \text{Int}(A^C)$.
- If $\forall U$-open s.t. $x \in U, U \cap A \neq \varnothing$ and $U \cap A^C \neq \varnothing$, then $x \in \partial A$, the boundary of A.

We put $\overline{A} := \text{Int}(A) \cup \partial A$, the closure of A. Equivalently, $x \in \overline{A} \iff$ for every open set $U : x \in U, U \cap A \neq \varnothing$. We call $x \in \overline{A}$ the limit points of A.

Remark 5. The limit point interpretation of the closure can be more intuitive; the points that we can get "arbitrary close to" are the closure. For instance, $\overline{(a,b)} = [a,b] \subseteq \mathbb{R}$ with the standard topology.

Proposition 2. *Let $A \subseteq X$-topological space. Then,* $\text{Int}(A)$ *is open, the largest open set contained in A, the union of all open sets contained in A, and* $\text{Int}(\text{Int}(A)) = \text{Int}(A)$. *Also, $\overline{A}$ closed, the smallest closed set that contains A, $\overline{A}$ the intersection of all closed sets that A is contained in, and $\overline{\overline{A}} = \overline{A}$.*

Corollary 1. *A open $\iff A = \text{Int}(A)$ and A closed $\iff A = \overline{A}$*

Remark 6. Remark that these are not exclusive, nor indeed the only possibilities.

Definition 7 (Basis)**.** A basis for a topology X with open sets τ is a collection $B \subseteq \tau$ such that every $U \in \tau$ a union of sets in B.

Remark 7. Don't think about bases for vector spaces in this regard - there is no "minimality" requirement.

Keep in mind $\{(a,b) : -\infty < a < b < \infty\}$, a basis of topology on $\mathbb{R}$.

Proposition 3. *For a metric space (X,d), $\{B(x,r) : x \in X, r > 0\}$ a basis of topology.*

Definition 8 (Subspace Topology)**.** For a subset $Y \subseteq X$-topological space, we define the subspace topology on Y as $\tau_Y := \{Y \cap U : U \in \tau\}$.

Definition 9 (★ Continuous)**.** For X, Y-topological spaces, a function $f : X \to Y$ is continuous iff $\forall V$-open in Y, $f^{-1}(V)$-open in X.

Remark 8. One can verify that this is consistent with the $\varepsilon - \delta$ definition of continuity for functions on $\mathbb{R}$.

Theorem 1 (Continuity of Composition)**.** *If $f : X \to Y$, $g : Y \to Z$ continuous, $g \circ f$ continuous.*

Remark 9. Note how much easier this is to prove via toplogical spaces than the $\varepsilon - \delta$ definition.

Definition 10 (Product Space)**.** For an index set I and $X_\alpha, \alpha \in I$, we define $\prod_{\alpha \in I} X_\alpha$ as a product space; I may be finite or infinite.

Proposition 4. *A basis for the product space is given by cyliders of the form* $A = \prod_{\alpha \in J} A_\alpha \times \prod \alpha \notin J X_\alpha$ *for* A_α*-open in* X_α*, where* $J \subseteq I$*-finite.*

Definition 11 (Compact)**.** A set $A \subseteq X$ is compact if every cover has a finite subcover, that is

$$A \subseteq \bigcup_{\alpha \in I} U_\alpha\text{-open} \implies \exists \{\alpha_1, \ldots, \alpha_n\} \subseteq I \text{ s.t. } A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Proposition 5. *Closed intervals* $[a, b]$ *compact in* $\mathbb{R}$*.*

Proposition 6. $A \subseteq \mathbb{R}^n$ *compact* $\iff$ *closed and bounded.*

Definition 12 (Connected)**.** X is said to not be connected if $X = U \cup V$ for U, V open, nonempty, disjoint. If X cannot be written as such, X is said to be connected.

Theorem 2. *If* X *connected and* $f : X \to Y$*, then* $f(X)$ *connected in* Y*.*

Proposition 7. *Intervals in* $\mathbb{R}$ *are connected.*

Theorem 3 (Intermediate Value Theorem)**.** *If* X *connected,* $f : X \to \mathbb{R}$ *continuous, then* f *takes intermediate value; if* $a = f(x), b = f(y)$ *for* $x, y \in X$ *with* $a < b$*, then for any* $a < c < b$ $\exists z \in X$ *s.t.* $f(z) = c$*.*

Theorem 4. *For* X *compact,* $f : X \to Y$ *continuous,* $f(X)$ *compact in* Y*.*

Proposition 8. *For* X *compact and* $f : X \to \mathbb{R}$*,* f *attains both max and min on* X*.*

Definition 13 (Path Connected)**.** A set $A \subseteq X$ is path connected if for any $x, y \in A, \exists f : [a, b] \to X$ continuous such that $f(a) = x, f(b) = y f([a, b]) \subseteq A$.

Theorem 5. *Path connected* $\implies$ *connected.*

For open sets in $\mathbb{R}^n$*, the converse holds too.*

Definition 14 (Connected Component, Path Component)**.** For $x \in X$, the connected component of x is the largest connected subset of X containing x and the path component of x is the largest path connected subset of X containing x.

2 Metric Spaces

We discuss mostly the metric on ℓ_p space and notions of completeness, as well as some topological results specific to metric spaces, namely compactness.

Definition 15 (ℓ_p)**.** For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $1 \leqslant p \leqslant +\infty$, we define

$$||x||_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}, \quad ||x||_\infty := \max_{i=1}^{n} |x_i| \, ,$$

and similarly, for sequences $x = (x_1, \dots, x_n, \dots)$,

$$||x||_p := \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}, \quad ||x||_\infty := \sup_{i=1}^{\infty} |x_i| \, ,$$

and define $\ell_p := \{x : ||x||_p < +\infty\}$. It can be shown that these are well-defined norms on their respective spaces.

Theorem 6 (Holder, Minkowski's Inequalities)**.** *For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ and p, q such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\textit{Holder's:} \qquad \langle x, y \rangle = \left| \sum_{i=1}^{n} x_i y_i \right| \leqslant ||x||_p ||y||_q$$

and

$$\textit{Minkowski's:} \qquad ||x + y||_p \leqslant ||x||_p + ||y||_p .$$

The identical inequalities hold for infinite sequences.

Definition 16 (Completeness)**.** We say a metric space is complete if every Cauchy sequence converges to a limit point in the space.

Proposition 9. *For $\{x_n\}_{n \in \mathbb{N}}$, ℓ_p complete for any $1 \leqslant p \leqslant +\infty$.*

Proposition 10. *If $p < q$, $\ell_p \subseteq \ell_q$.*

Definition 17 (Contraction Mapping)**.** For a metric space (X, d), a function $f : X \to X$ is a contraction mapping if there exists $0 < c < 1$ such that

$$d(f(x), f(y)) \leqslant c \cdot d(x, y)$$

for any $x, y \in X$.

Theorem 7. *Let (X, d) be a complete metric space, $f : X \to X$ a contraction. Then, there exist a unique fixed point z of f such that $f(z) = z$; ie $\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f \circ f \circ \cdots \circ f(x) = z$ for any $x \in X$.*

Theorem 8. *ℓ_p complete.*

Remark 10. It can be kind of funky to work with sequences in ℓ_p, since the elements of ℓ_p themselves sequences so we have "sequences of sequences".

Definition 18 (Totally bounded)**.** A metric space X is said to be totally bounded if $\forall\, \varepsilon > 0 \exists x_1, \ldots, x_n \in X, n = n(\varepsilon)$ such that $\bigcup_{i=1}^{n} B(x_i, \varepsilon) = X$.

Definition 19 (Sequentially compact)**.** A metric space X is said to be sequentially compact if every sequence has a convergent subsequence.

Theorem 9 (★ Equivalent Notions of Compactness in Metric Spaces)**.** *Let (X, d) a metric space. TFAE:*

- *X compact*
- *X complete and totally bounded*
- *X sequentially compact*

Remark 11. This is for a metric space, not a general topological space! Hopefully this is clear because some of the requirements necessitate a distance.

3 Differentiation

Definition 20 (Differentiable)**.** $f(x)$ differentiable at c if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, and if so we denote the limit $f'(c)$.

Alternatively, one can view differentiation as a linear map between spaces of differentiable functions.

Theorem 10. *Differentiable* $\implies$ *continuous.*

Proof. Short enough to write the full proof; $\lim_{x\to c}(f(x)-f(c)) = \lim_{x\to c}(x-c)\frac{f(x)-f(c)}{x-c} = 0 \cdot f'(c) = 0$. □

Theorem 11 (Caratheodory's)**.** *For* $f : I \to \mathbb{R}, c \in I$*,* f *differentiable at* c *iff* $\exists \varphi : I \to \mathbb{R} : \varphi$ *continuous at* c*,* $f(x) - f(c) = \varphi(x)(x-c)$*.*

Sketch. Its worth recalling the definition of φ for the forward implication,

$$\varphi(x) := \begin{cases} \frac{f(x)-f(c)}{x-c} & x \neq c \\ f'(c) & x = c \end{cases}.$$

The converse follows by taking limits. □

Remark 12. While not a particularly enlightening result, used in proofs of the chain rule, etc.

Theorem 12 (Chain Rule)**.** *Let* $f : J \to \mathbb{R}, g : I \to R$ *s.t.* $f(J) \subseteq I$*. If* $f(x)$ *differentiable at* c *and* $g(y)$ *at* $f(c)$*,* $g \circ f$ *differentiable at* c *with* $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$*.*

Sketch. Apply Caratheodory's to f at c and g at $f(c)$, and compose. □

Theorem 13 (Rolle's)**.** *Let* $f : [a,b] \to \mathbb{R}$ *continuous. If* $f'(x)$ *exists on* (a,b) *and* $f(a) = f(b) = 0$*,* $\exists c \in (a,b) : f'(c) = 0$*.*

Sketch. If constant function, done. Else, assuming function positive, it obtains a maximum, and thus its derivative 0 at this point. □

Theorem 14 (★ Mean Value)**.** *Let f continuous on $[a,b]$ and differentiable on (a,b). Then, $\exists c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.*

Sketch. Let $\phi(x) := f(x) - f(a) - \frac{f(b)-f(a)}{(b-a)}(x-a)$. Then $\phi(a) = \phi(b) = 0$ so applying Rolle's $\exists c \in (a,b) : \varphi'(c) = 0 = f'(x) - \frac{f(b)-f(a)}{b-a}$. The proof is done after rearranging. □

Proposition 11 (L'Hopital's)**.** *If $f, g : [a,b] \to \mathbb{R}$ with $f(a) = g(a) = 0$, $g(x) \neq 0$ on $a < x < b$, f, g differentiable at $x = 0$ with $g'(a) \neq 0$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)}$ exists and is equal to $\frac{f'(a)}{g'(a)}$.*

Remark 13. Other versions exist, but this is certainly one of them.

Theorem 15 (★ Taylor's)**.** *Let $f \in C^n([a,b])$ such that $f^{(n+1)}(x)$ exists on (a,b). Let $x_0 \in [a,b]$, then, for any $x \in [a,b]$, $\exists c$ between x, x_0 such that*

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

Corollary 2. *Let $x_0 \in [a,b]$. With the same assumptions as Taylor's (but in a neighborhood of x_0), with $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$, then*

- *n even; then f has a local minimum at x_0 if $f^{(n)}(x_0) > 0$ and a local max if $f^{(n)}(x_0) < 0$.*
- *n odd; neither.*

4 Integration

Its all just rectangles.

Definition 21 (Riemann Integration)**.** Consider an interval (a,b). We call a subdivision $\mathcal{P} := \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$ a partition, and $\dot{\mathcal{P}}$ a marked partition if in addition we are given a point $t_i \in (x_i, x_{i+1}]$ for each interval in $\dot{\mathcal{P}}$.

We put $\operatorname{diam}(\mathcal{P}) := \max_{i=1}^{n} |x_i - x_{i-1}|$.

We define the Riemann sum $S(f, \dot{\mathcal{P}}) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$, and say that f Riemann integrable on $[a,b]$ if $S(f, \dot{\mathcal{P}}) \to L$ as $\operatorname{diam}(\dot{\mathcal{P}}) \to 0$ for any choice of tag t_i, and write $f \in \mathcal{R}([a,b])$

More precisely, if $\forall \varepsilon > 0, \exists \delta > 0 : \operatorname{diam}(\mathcal{P}) < \delta$, then for any $t_i \in [x_i, x_{i+1}]$, $\left|L - S(f, \dot{\mathcal{P}})\right| < \varepsilon$. We then say the (Riemann) integral of f over $[a, b]$ is L and write $\int_a^b f(x)\,\mathrm{d}x = L$.

Proposition 12. *Riemann integrals are unique, linear in $f(x)$, and respect inequalities (if $f \leqslant g$ on $[a, b]$, $\int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x$ if both in $\mathcal{R}([a, b])$)*

Proposition 13 (★)**.** *$f \in \mathcal{R}[a, b] \implies f$ bounded on $[a, b]$*

Proposition 14 (★ Cauchy Criterion for Integrability)**.** *$f \in \mathcal{R}[a, b] \iff \forall \varepsilon > 0, \exists \delta > 0 :$ if $\dot{P}$ and $\dot{Q}$ are tagged partitions of $[a, b]$ s.t. $\operatorname{diam} \dot{P} < \delta$ and $\operatorname{diam} \dot{Q} < \delta$, then $\left|S(f, \dot{P}) - S(f, \dot{Q})\right| < \varepsilon$*

Remark 14. Ala Cauchy Sequence.

Theorem 16 (Squeeze Theorem)**.** *$f \in \mathcal{R}[a, b] \iff \forall \varepsilon > 0, \exists \alpha_\varepsilon, \omega_\varepsilon \in \mathcal{R}[a, b] : \alpha_\varepsilon \leqslant f \leqslant \omega_\varepsilon$ and $\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$.*

Lemma 1. *Let $J := [c, d] \subseteq [a, b]$ and $\varphi_J(x) := \begin{cases} 1 & x \in J \\ 0 & x \notin J \end{cases}$ be the indicator function of J. Then $\varphi_J \in \mathcal{R}[a, b]$ and $\int_a^b \varphi_J = d - c$.*

Remark 15. Helpful for "approximations"; follows by linearity, induction that step functions (ie sums of indicator functions times constants) are integrable.

Theorem 17 (★ Continuous)**.** *f continuous on $[a, b] \implies f \in \mathcal{R}[a, b]$*

Sketch. Continuity on a closed interval gives uniform continuity and so a "universal δ"; then, for any partition, take the x such that f attains its minimum and maximum, and define a α_ε, ω_ε as the sum of indicator functions taking the minimum, maximum of f respectively on each partition. Then apply the previous theorem and the squeeze theorem. □

Theorem 18 (Additivity)**.** *$f \in \mathcal{R}[a, b] \iff f \in \mathcal{R}[a, c]$ and $f \in \mathcal{R}[c, b]$, and $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$.*

Theorem 19 (★ Fundamental Theorem of Calculus). *Let $F, f : [a,b] \to \mathbb{R}$ and $E \subseteq [a,b]$ a finite set, such that F continuous on $[a,b], F'(x) = f(x) \, \forall \, x \in [a,b] \setminus E$, $f \in \mathcal{R}[a,b]$. Then $\int_a^b f(x) = F(b) - F(a)$. We call F the "primitive" of f.*

Theorem 20. *For $f \in \mathcal{R}[a,b]$ and any $z \in [a,b]$, put $F(z) := \int_a^z f(x) \, \mathrm{d}X$. Then, F continuous on $[a,b]$.*

Theorem 21 (★ Fundamental Theorem of Calculus p2). *For $f \in \mathcal{R}[a,b]$ continuous at c, then $F(z)$ differentiable at c and $F'(c) = f(c)$.*

Definition 22 (Lebesgue Measure). We say a set $A \subseteq \mathbb{R}$ has Lebesgue measure 0 iff $\forall \, \varepsilon > 0$, A can be covered by a union of intervals J_k such that $\sum_k |J_k| \leqslant \varepsilon$. We then call A a "null set".

In particular, any countable set is a null set.

Theorem 22 (★ Lebesgue Integrability Criterion). *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}[a,b] \iff$ the set of discontinuities of f has Lebesgue measure 0.*

Remark 16. In particular, remark that continuity a stronger requirement than integrability.

Theorem 23 (Composition). *If $f \in \mathcal{R}[a,b]$, $\varphi : [c,d] \to \mathbb{R}$ continuous and $f([a,b]) \subseteq [c,d]$, then $\varphi \circ f \in \mathcal{R}[a,b]$.*

Theorem 24 (Integration by Parts). *If F, G differentiable $[a,b]$ with $f := F', g := G'$, and $f, g \in \mathcal{R}[a,b]$, then*

$$\int_a^b f(x)G(x) \, \mathrm{d}x = F(x)G(x)\Big|_a^b - \int_a^b F(x)g(x) \, \mathrm{d}x \, .$$

Sketch. Uses additivity and the fundamental theorem of calculus. □

Theorem 25 (Taylor's Theorem, Remainder's Version). *Suppose $f', f'', \ldots, f^{(n)}$ exist on $[a,b]$ and $f^{(n+1)} \in \mathcal{R}[a,b]$. Then*

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n,$$

where $R_n := \frac{1}{n!}\int_a^b f^{(n+1)}(t)(b-t)^n \, \mathrm{d}t$.

5 Sequences of Functions

A good motivation to keep in mind with the "types" of function-sequence convergence is to answer the question: when can we exchange limits of derivatives of functions and derivatives of limits of functions? What about integrals? What about summations (see next section)? Ie, when does $\lim_{n\to\infty} f_n'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\lim_{n\to\infty} f_n(x)$, etc.

Definition 23 (Pointwise, Uniform Convergence)**.** We say $f_n \to f$ pointwise on E if $\forall\, x \in E$, $f_n(x) \to f(x)$ as $n \to \infty$.

We say $f_n \to f$ uniformly on E if $\forall\, \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall\, n \geqslant N, x \in E$, $|f_n(x) - f(x)| < \varepsilon$.

Remark 17. Pointwise doesn't care about the "rate of convergence"; as long as each point converges eventually, we're good. Uniform convergence needs all points to converge "at the same rate" (so to speak).

A good example to keep in mind is $f_n := \begin{cases} 2nx & 0 \leqslant x \leqslant \frac{1}{2n} \\ 0 & x > \frac{1}{2n} \end{cases}$ on $[0,1]$, which converges pointwise to 0 but not uniformly.

A good trick for disproving uniform convergence of $f_n \to f$ is by showing $f_n(x_0)$ constant and $\neq f(x_0)$ for all n. For instance, $f_n(x) := \sin\left(\frac{x}{n}\right) \to 0$ pointwise, but $f_n(\frac{n\pi}{2}) = 1 \,\forall\, n$ so the convergence os not uniform.

Proposition 15. *Uniform $\implies$ pointwise convergence.*

Theorem 26. *The metric space of continuous functions $C([a,b])$ complete with respect to $d_\infty(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|$.*

Theorem 27 (★ Interchange of Limits)**.** *Let $J \subseteq \mathbb{R}$ be a bounded interval such that $\exists x_0 \in J$: $f_n(x_0) \to f(x_0)$. Suppose $f_n'(x) \to g(x)$ uniformly on J. Then, $\exists f : f_n(x) \to f(x)$ uniformly on J, $f(x)$ differentiable on J, and moreover $f_n'(x) = g(x) \,\forall\, x \in J$.*

Theorem 28 (★ Interchange of Integrals). *Let $f_n \in \mathcal{R}[a,b]$, $f_n \to f$ uniformly on $[a,b]$. Then $f \in \mathcal{R}[a,b]$ and $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)\,\mathrm{d}x$*

Theorem 29 (Bounded Convergence). *Let $f_n \in \mathcal{R}[a,b]$, $f_n \to f \in \mathcal{R}[a,b]$ (not necessarily uniform). Suppose $\exists B > 0$ s.t. $|f_n(x)| \leqslant B \,\forall\, x \in [a,b]$ and $\forall\, n \in \mathbb{N}$, then $\int_a^b f_n \to \int_a^b f$ as $n \to \infty$.*

Remark 18. This provides a weaker condition, but equivalent result as the previous theorem, although remark now that we need the limit function itself to be in $\mathcal{R}[a,b]$, which was a result, not a necessity, of the previous theorem. In general, uniform continuity very strong, but leads to helpful results.

Theorem 30 (Dimi's). *If $f_n \in C([a,b])$, $f_n(x)$ monotone (as a sequence), and $f_n \to f \in C([a,b])$, then $f_n \to f$ uniformly.*

6 Infinite Series

Definition 24 (Covergence of Series). Let $\{x_j\} \in X$-normed vector space over $\mathbb{R}$. We say $\sum_{j=1}^{\infty} x_j$ converges absolutely iff $\sum_{j=1}^{\infty} ||x_j|| < +\infty$. In particular, if $X = \mathbb{R}$, then $|| \cdot || = |\cdot|$.

We say $\sum_{j=1}^{\infty} x_j$ converges conditionally if $\sum_{j=1}^{\infty} x_j < +\infty$, but $\sum_{j=1}^{\infty} ||x_j|| = +\infty$.

Proposition 16. *Any rearrangement of an absolutely convergent series gives the same sum. Conversely, the order of summation of a conditionally convergent summation can be rearranged such as to equal any real number.*

Proposition 17 (Absolute Convergence Tests).

- ***Comparison Test:*** *let x_n, y_n be nonzero real sequences and $r := \lim \left|\frac{x_n}{y_n}\right|$. If such a limit exists, then if*
  - *(a) $r \neq 0$, $\sum_n x_n$ absolutely convergent $\iff \sum_n y_n$ absolutely convergent.*
  - *(b) $r = 0$, $\sum_n y_n$ absoltuely convergent $\implies \sum_n x_n$ absolteuly convergent.*
- ***Root Test:*** *if $\exists r < 1$ s.t. $|x_n|^{1/n} \leqslant r \,\forall\, n \geqslant K$-sufficiently large, then $\sum_{n=K}^{\infty} |x_n|$ converges. Conversely, if$|x_n|^{1/n} \geqslant 1$ for $n \geqslant K$-sufficiently large, $\sum_n x_n$ diverges.*

- ***Ratio Test:*** *if $x_n \neq 0$ and $\exists 0 < r < 1$ s.t. $\left|\frac{x_{n+1}}{x_n}\right| \leqslant r$ for $n \geqslant K$ sufficiently large, $\sum_n x_n$ absolutely convergent. Conversely, if $\left|\frac{x_{n+1}}{x_n}\right| \geqslant 1$ for $n \geqslant K$ sufficiently large, then $\sum_n x_n$ diverges.*

- ***Integral Test:*** *if $f(x) \geqslant 0$ non-increasing/non-decreasing function of $x \geqslant 1$, $\sum_{k=1}^{\infty} f(k)$ converges iff $\lim_{k\to\infty} \int_1^k f(x)\,\mathrm{d}x$ finite.*

* ***Raube's Test:*** *let $x_n \neq 0$.*

  (a) If $\exists a > 1$ s.t. $\left|\frac{x_{n+1}}{x_n}\right| \leqslant 1 - \frac{1}{n}\,\forall n \geqslant K$-sufficiently large, then $\sum_n x_n$ converges absolutely.

  (b) If $\exists a \leqslant 1$ s.t. $\left|\frac{x_{n+1}}{x_n}\right| \geqslant 1 - \frac{1}{n}\,\forall n \geqslant K$-sufficiently large, $\sum_n x_n$ does not converge absolutely.

Remark 19. Proofs of these tests aren't really important (Dima-speaking), but knowing the conditions in which they apply is.

Proposition 18 (Tests for Non-Absolute Convergence)**.**

- ***Alternating Series:*** *if $x > 0$, $x_{n+1} \leqslant x_n$ such that $\lim_{n\to\infty} x_n = 0$, then $\sum_n (-1)^n x_n$ converges.*

- ***Dirichlet's Test:*** *if x_n decreasing with limit 0, and the partial sum $s_n := y_1 + \cdots + y_n$ is bounded, then $\sum_n x_n y_n$ converges.*

- ***Abel's Test:*** *let x_n convergent and monotone, and suppose $\sum_n y_n$ converges. Then $\sum_n x_n y_n$ also converges.*

Definition 25 (Convergence of Series of Functions)**.** We say a series $\sum_n f_n(x)$ converges absolutely to some $g(x)$ on E if $\sum_n |f_n(x)|$ converges for all $x \in E$.

We say that the convergence is uniform if it is uniform for any $x \in E$, ie $\forall\, \varepsilon > 0 \exists N \in \mathbb{N}$ s.t. $\forall\, n \geqslant N, x \in E, |g(x) - \sum_n f_n(x)| < \varepsilon$.

Proposition 19 (Interchanging Integrals and Summations)**.** *Suppose for $f_n : [a,b] \to \mathbb{R}$, $\sum_n f_n(x) \to g(x)$ uniformly and $f_n \in \mathcal{R}[a,b]$. Then $\int_a^b g(x) = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,\mathrm{d}x$.*

Proposition 20 (Interchanging Derivatives and Summations)**.** *Let $f_n : [a,b] \to \mathbb{R}$, $f_n' \exists$, $\sum_n f(x)$ converges for some $[a,b]$ and $\sum_n f_n'(x)$ converges uniformly. Then, there exists some $g : [a,b] \to \mathbb{R}$ such that $\sum_n f_n \to g$ uniformly, g differentiable, and $g'(x) = \sum_n f_n'(x)$, all on $[a,b]$.*

Theorem 31 (⋆ Cauchy Criterion of Series)**.** *$f_n(x) : D \to \mathbb{R}$ converges uniformly on D iff $\forall\, \varepsilon > 0, \exists N$ s.t. $\forall\, m,n \geqslant N, \sum_{i=n+1}^{m} f_i(x) < \varepsilon \,\forall\, x \in D$.*

Proposition 21 (Weierstrass M-Test)**.** *If $|f_n(x)| \leqslant M_n \,\forall\, x \in D \subseteq \mathbb{R}$ and $\sum_n M_n < +\infty$, then $\sum_n f_n(x)$ converges uniformly on D.*

Definition 26 (Power Series)**.** A function of the form $f(x) \coloneqq \sum_{n=0}^{\infty} a_n(x-c)^n$ is said to be a power series centered at c.

Put $\rho \coloneqq \limsup_{n\to\infty} \sqrt[n]{|a_n|}$, and put

$$R \coloneqq \begin{cases} \frac{1}{\rho} & 0 < \rho < +\infty \\ 0 & \rho = +\infty \\ \infty & \rho = 0 \end{cases}.$$

We call R the radius of convergence of f.

Theorem 32 (⋆ Cauchy-Hadamard)**.** *Let R be the radius of converges of f. Then, $f(x)$ converges if $|x-c| < R$, and diverges if $|x-c| > R$.*

Sketch. Apply the root test to the definition of R. □

Remark 20. If $|x-c| = R$, the theorem is inconclusive, and we need to manually check.