FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 255

Honours Analysis 2

Examiner: Professor I. Klemes
Associate Examiner: Professor D. Jakobson

Date: Friday, 13 April, 2007
Time: 2pm-5pm

INSTRUCTIONS

Calculators are not permitted.
This is a closed book examination.
Answer all 6 questions, in the examination booklets.
Keep this exam paper.

This exam comprises the cover and one page of questions, PRINTED Double-Sided

1. (10 points) If f is a bounded function on $[a, b]$ and P and Q are partitions of $[a, b]$, prove that there is a partition R of $[a, b]$ such that R satisfies both of the following inequalities:
$$L(P, f) \leq L(R, f) \quad \text{and} \quad U(R, f) \leq U(Q, f).$$

2. (15 points) Let $f : [a, b] \to \mathbb{R}$ be a bounded function.
   (a) Define the oscillation $\omega_f(x)$ of f at a point x.
   (b) Prove that f is continuous at x_0 if and only if $\omega_f(x_0) = 0$.

3. (15 points)
   (a) Define the concept "set of measure zero" and prove that any subset of a set of measure zero is a set of measure zero.
   (b) State Lebesgue's Criterion for Riemann integrability.
   (c) Let $f : [a, b] \to [0, \infty)$ be an integrable function such that $\{x \in [a, b] \mid f(x) > 0\}$ is not a set of measure zero. Prove that $\int_a^b f > 0$.

4. (10 points)
   (a) Give an example of a continuous $F : [0, 1] \to \mathbb{R}$ such that $F'(x)$ exists at all points $x \in [0, 1]$ except $x = \frac{1}{2}$.
   (b) If $F : [0, 1] \to \mathbb{R}$ is continuous, $f : [0, 1] \to \mathbb{R}$ is integrable, and $F'(x) = f(x)$ at all points $x \in [0, 1]$ except $x = \frac{1}{3}$ and $x = \frac{2}{3}$, prove that $\int_0^1 f = F(1) - F(0)$. (You may assume the first fundamental theorem of calculus.)

5. (10 points)
   (a) Define "open set" and "closed set" in $\mathbb{R}$.
   (b) Show that the union of any family of open sets is open and that the intersection of any family of closed sets is closed.

6. (10 points)
   (a) State the Weierstrass M-Test.
   (b) Fix $r > 0$ and let $D = [r, \infty)$. Prove that the function f defined by
$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(3^n x)}{n^2 x^2 + 1}, \qquad (x \in D)$$
   is continuous on D.