McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 255

ANALYSIS 2

Examiner: Professor J. Labute
Associate Examiner: Professor R. Vermes

Date: Wednesday, April 16, 2003
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

Do any SIX questions.
Justify all your statements.
No calculators allowed.

This exam comprises the cover and 1 page with 8 questions.

1. Give the definition of Riemann integrability and show that a Riemann integrable function is bounded. Show that a function is Riemann integrable if and only if it satisfies Riemann's condition.

2. Show how the Riemann-Stieltjes integral can be used to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{s}{s-1} - s\int_1^{\infty} \frac{((x))}{x^{s+1}}\, dx,$$

where $((x)) = x - [x]$ is the fractional part of x. Show that the series and improper integral converge absolutely and uniformly for $s \geq 1+\epsilon$ and $s \geq \epsilon$ respectively for any $\epsilon > 0$. What can you deduce about the infinite series and improper integral as functions of s?

3. (a) Prove that the set of discontinuities of a Riemann integrable function is of measure zero.
   (b) If g is Riemann integrable on $[a, b]$ with $m \leq g(x) \leq M$ on $[a, b]$ and f continuous on $[m, M]$, prove that $h(x) = f(g(x))$ is Riemann integrable.

4. (a) If $a > 0$, show that the sequence $(n^2x^2e^{-nx})_{n\geq 1}$ converges uniformly on $[a, \infty)$ but does not converge uniformly on $[0, \infty)$.
   (b) Find a sequence of continuous functions f_n on $[0, 1]$ which converges to a continuous function f on $[0, 1]$ such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx \neq \int_0^1 f(x)\, dx.$$

5. Show that the series

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

converges uniformly on $\mathbb{R}$ and that

$$f'(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n}$$

if x is not an integral multiple of 2π. What can you say about the continuity of f'?

6. Using the Taylor series expansion of $(1 - x^2)^{-1/2}$ about $x = 0$, show that

$$\sin^{-1}(x) = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\frac{x^{2n+1}}{2n+1} + \cdots.$$

for $|x| \leq 1$. What is the radius of convergence of this series? Using a suitable estimate for the remainder of the series for $\sin^{-1}(1)$, estimate how many terms of this series would be sufficient to compute π to three decimal places. How many terms would be sufficient if the series for $\sin^{-1}(1/2)$ were used?

7. Define what is meant by compact subset of a metric space. If $f : X \to Y$ is a continuous mapping of metric spaces with X compact, show that f is uniformly continuous. Show also that $f(X)$ is compact.

8. Let $S \subset \mathbb{R}$ be compact and let X be the set of continuous real-valued functions on S. Show that X is a metric space with metric

$$d(f, g) = \sup_{x\in S} |f(x) - g(x)|$$

and that this metric space is complete.