1. (i) (4 marks) Define the term *metric space.*
   (ii) (4 marks) Define the term *open subset* of a metric space.
   (iii) (4 marks) Define the term *closed subset* of a metric space.
   (iv) (8 marks) Show from first principles that a subset of a metric space is closed if and only if its complement is open.

2. For each of the following series, determine whether the series converges. Justify your answer.
   (i) (5 marks) $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right).$
   (ii) (5 marks) $\displaystyle\sum_{n=1}^{\infty}\sin\left(\pi\frac{n^2+1}{n}\right).$
   (iii) (5 marks) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n\, n!}{n^n}.$
   (iv) (5 marks) $\displaystyle\sum_{n=3}^{\infty}(\ln n)^{-\ln n}.$

3. (i) (4 marks) Define the term *Riemann partition.*
   (ii) (4 marks) Define the upper and lower Riemann sums $U(P,f)$ and $L(P,f)$ for a Riemann partition P.

   Let $f:[0,1]\longrightarrow[-1,1]$ be defined by $f(x)=(-1)^k$ if $x\in]2^{-(k+1)},2^{-k}]$ and $f(0)=0$.

   (iii) (8 marks) Given $\epsilon>0$ find explicitly a Riemann partition P of $[0,1]$ with $U(P,f)-L(P,f)<\epsilon$. Justify your answer. What is the significance of what you have just shown?
   (iv) (4 marks) What is the value of $\int_0^1 f(x)dx$?

4. For each of the following sequences of functions defined on $]0,\infty[$ determine the pointwise limit. Determine also whether convergence is uniform on $]0,\infty[$. Justify your answer.
   (i) (10 marks) $\displaystyle f_n(x)=\frac{\lceil nx\rceil}{n\lceil x\rceil}.$

   Note: The notation $\lceil x\rceil$ denotes the unique *integer* k such that $k-1<x\le k$.

   (ii) (10 marks) $\displaystyle f_n(x)=\frac{\sin(nx)}{nx}.$

5. (20 marks) Consider the power series

$$f(x) = x - \frac{3}{4}\,x^4 + \frac{3^2}{4\cdot 7}\,x^7 - \frac{3^3}{4\cdot 7\cdot 10}\,x^{10} + \frac{3^4}{4\cdot 7\cdot 10\cdot 13}\,x^{13} - \cdots$$

(i) (4 marks) What is the radius of convergence ρ of this series?

(ii) (6 marks) Show that $f'(x) + 3x^2 f(x) = 1$ for $|x| < \rho$. Outline briefly the theorems that you are using.

(iii) (6 marks) Let $g(x) = e^{-x^3}\displaystyle\int_{u=0}^{x} e^{u^3}\,du$. Show that $g'(x) + 3x^2 g(x) = 1$ for all real x. Outline briefly the theorems that you are using.

(iv) (4 marks) Deduce that $f(x) = g(x)$ for $|x| < \rho$.

6. (i) (4 marks) If $p > 0$ and n is a nonnegative integer, show that the function $x \mapsto x^p\Big(-\ln(x)\Big)^n$ is continuous on $[0,1]$.

(ii) (4 marks) Show that the series $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}\Big(-x\ln(x)\Big)^n$ converges uniformly on $[0,1]$.

(iii) (4 marks) State a theorem about the integral of a uniform limit.

(iv) (2 marks) Show that $\displaystyle\int_0^1 x^{-x}\,dx = \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1\Big(-x\ln(x)\Big)^n dx$.

(v) (4 marks) Let $p > 0$ and let n be a nonnegative integer. Show by induction that $\displaystyle\int_0^1 x^p\Big(-\ln(x)\Big)^n dx = \frac{n!}{(p+1)^{n+1}}$.

(vi) (2 marks) Deduce that $\displaystyle\int_0^1 x^{-x}\,dx = \sum_{n=0}^{\infty}(n+1)^{-(n+1)}$.

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FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-255B

Analysis II

Examiner: Professor S. W. Drury
Associate Examiner: Professor K. N. GowriSankaran

Date: Monday, April 23
Time: 2: 00 pm. – 5: 00 pm.

INSTRUCTIONS

All six questions should be attempted for full credit.

This is a closed book examination.
Write your answers in the booklets provided.
No calculators are allowed.

All questions are of equal weight; each is worth 20 marks.
The exam will be marked out of a total of 120 marks
and subsequently scaled to a percentage.

This exam comprises the cover and 2 pages of questions.