McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 251

HONOURS ALGEBRA 2

Examiner: Professor H.Darmon
Associate Examiner: Professor O. Kharlampovich

Date: Tuesday April 18, 2006
Time: 2:00 PM - 5:00 PM

INSTRUCTIONS

1. Please answer all questions in the exam booklets provide.
2. This is a closed book exam. No notes or texts allowed.
3. Calculators are not permitted.
4. Use of a regular or translation dictionary is not permitted.

This exam comprises the cover page, and 3 pages of 7 questions.

MATH 251: Algebra 2

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1. Let $T : V \longrightarrow W$ be a surjective linear transformation of vector spaces over a field F, and suppose that W is finite-dimensional.

(a) Show that there is a subspace $U \subset V$ with the property that the restriction $T|_U : U \longrightarrow W$ is an isomorphism between U and W.

(b) Show that $V = U \oplus \ker(T)$.

(c) Prove or disprove: there is only one subspace U of V satisfying (b).

2. Let V be a finite-dimensional real vector space and let $T : V \longrightarrow V$ be a linear transformation.

(a) Show that if $\dim(V)$ is odd, then T has an eigenvector.

(b) Give an example to show that this is not true in general when $\dim(V)$ is even.

3. Let $T : V_1 \longrightarrow V_2$ be a linear transformation between finite-dimensional vector spaces, and let $T^* : V_2^* \longrightarrow V_1^*$ denote the resulting linear transformation on the dual spaces. (Recall from the class that T^* is defined by $T^*(\ell) = \ell \circ T$, for all $\ell \in V_2^*$.)

(a) Show that if T is surjective then T^* is injective.

(b) Show that if T is injective then T^* is surjective.

4. Let V be a vector space of dimension n over a field F, and let $T : V \longrightarrow V$ be a linear transformation whose minimal polynomial has degree n and is *irreducible* over F.

(a) Show that the subring of $\mathcal{L}(V, V)$ defined by

$$K = \{g(T), \quad \text{with } g \in F[x]\}$$

is a field which contains F. What is its dimension as a vector space over F?

(b) Show that the set V becomes a vector space over K with the scalar multiplication defined by

$$(f(T), v) \mapsto f(T)(v).$$

What is the dimension of V over K?

(c) Let $U : V \longrightarrow V$ be a linear transformation of vector spaces over F which commutes with T. Show that U is also linear with respect to the scalar multiplication by K, i.e., it is a linear transformation on V viewed as a vector space over K.

(d) Use (c) to show that any linear transformation U that commutes with T can be expressed as a polynomial in T.

5. Let V be an inner product space over $\mathbf{R}$ and let $T : V \longrightarrow V$ be a self-adjoint transformation on V.

(a) State (without proof) what the spectral theorem tells us about T.

(b) Let $U \subset V$ be a subspace of V which is stable under T (i.e., $T(U) \subset U$.) Show that U has a complementary T-stable subspace, i.e., a subspace $W \subset V$ such that $T(W) \subset W$ and $V = U \oplus W$.

(c) Give an example to show that the statement of (b) can be false when T is not self-adjoint.

6. Find the linear function $f(x) = ax + b$ which minimizes the quantity

$$\int_0^{\pi} (f(t) - \sin(t))^2 dt$$

7. True or false? (You do not need to justify your answer.)

(a) If $T : V \longrightarrow W$ is a linear transformation, and V is a finite-dimensional vector space, then

$$\dim(\ker(T)) + \dim(\mathrm{Im}(T)) = \dim(V).$$

(b) If $T : V \longrightarrow V$ is a linear transformation, and V is a finite-dimensional vector space, then

$$\ker(T) \oplus \mathrm{Im}(T) = V.$$

(c) Every invertible linear transformation on a finite-dimensional complex vector space has a square root.

(d) The characteristic and minimal polynomials of a linear transformation acting on a finite-dimensional complex vector space have exactly the same roots.

(e) A linear transformation is diagonalisable if its characteristic polynomial factors into distinct linear factors.

(f) A linear transformation is diagonalisable if and only if its minimal polynomial factors into distinct linear factors.