Final Examination December 6, 1996 Mathematics 189-249A

MARKS

(9) 1. Consider an entire function $f(z)$ for which $\Im[f'(z)] = -e^y \sin x$, $f(0) = 5 + 3i$, $f(\pi/2) = 2i$. Express $f(z)$ as a function of $z$.

(10) 2. Evaluate

$$\int_C \frac{dz}{z(e^z - 1)}$$

where $C$ is the circle $|z - 3i| = 12$.

(12) 3. Evaluate

$$\int_0^\infty \frac{\sin^2 x}{x^2(x^2+1)} dx \ .$$

Justify carefully your conclusions.

4. Find the inverse Laplace transforms of

(9) (a) $\dfrac{e^{-\beta s^{1/2}}}{s}$, $\beta > 0$. Express your answer in terms of an error function. (See useful information.)

(11) (b) $\dfrac{1}{s^3 \sinh s}$, expressing your answer in real form.

(12) 5. Find the sums of the following series using residues:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

(6) 6. Evaluate

(a) $\dfrac{y}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{\cos \xi d\xi}{(x-\xi)^2 + y^2}$, and (b) $\dfrac{y}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{\sin \xi d\xi}{(x-\xi)^2 + y^2}$.

(18) 7. (a) Use the Schwarz-Christoffel transformation to determine a function that maps the region indicated in the $w$-plane onto the upper half of the $z$-plane with the boundary mapped as shown.

(b) Find the potential (steady-state temperature) distribution in the semi-infinite slab of 13 units whose boundaries are maintained at the potentials (steady-state temperatures) indicated.

(c) Show that if $f(z) = u(x,y) + iv(x,y)$ is an analytic function of a complex variable then

$$R[\sin^{-1} f(z)] = \sin^{-1}\left[\frac{\sqrt{(u+1)^2+v^2} - \sqrt{(u-1)^2+v^2}}{2}\right] .$$

(d) Find the steady-state distribution in the region below.

If $z = re^{i\theta}$, express your final answer in terms of $r$ and $\theta$.

(3) 8. (a) Deduce the complex potential for a uniform line source of strength $k$ at the origin.

(3) (b) Deduce the complex potential for a vortex at the origin with circulation $\gamma$.

(7) (c) Obtain the complex potential for flow around a long circular cylinder of unit radius immersed in a fluid moving uniformly with speed $V_0$ parallel to the $x$-axis.

Write down the potential function, stream function, stagnation points and sketch the flow pattern.

Useful Information

1. $\displaystyle\int_0^\infty e^{-ax^2}\cos bx dx = \sqrt{\frac{\pi}{4a}}e^{-b^2/4a},\ a > 0.$

2. $\cos 2\theta = 1 - 2\sin^2\theta.$

3. $\displaystyle\int \frac{dz}{\sqrt{1-z^2}} = \sin^{-1} z + C.$

4. $\displaystyle\int \frac{dz}{\sqrt{z^2-1}} = \cosh^{-1} z + C.$

5. $\displaystyle \text{erf}x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du.$

Good Luck!

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-249A

ADVANCED CALCULUS II

Examiner: Professor C. Roth
Associate Examiner: Professor S. Melamed

Date: Friday, December 6, 1996
Time: 2:00 P.M. - 6:00 P.M.

INSTRUCTIONS

**Calculators are not permitted**

**NOTE: Express all your answers in the form $a + bi$ with $a$ and $b$ real**

This exam comprises the cover and 3 pages of questions.