FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS MATH 242

Analysis 1

Examiner: Professor S. W. Drury

Associate Examiner: Professor V. Jaksic

Date: Friday, 18 December 2009

Time: 9: 00 am. – 12: 00 noon.

INSTRUCTIONS

Answer all questions in the booklets provided.
This is a closed book examination.
Calculators are not permitted.
Both regular and translation dictionaries are allowed.

Read the questions carefully before answering them.

In questions 1-4 inclusive all proofs are to be given from first principles. Only the most basic properties of real numbers may be assumed. Proofs must be self-contained and you must work directly from the definitions of the concepts involved.

For all questions, write your answer in a clear, complete and logical way. Do not introduce unnecessary ideas.

This exam has 9 questions and 3 pages

1. (6 points) If $x_n \in \mathbb{R}$, $x_n \underset{n\to\infty}{\longrightarrow} a$ and $x_n \underset{n\to\infty}{\longrightarrow} b$, then show from first principles that $a = b$.

2. (6 points) Let $x_n \in \mathbb{Z}$, $x \in \mathbb{R}$ and $x_n \underset{n\to\infty}{\longrightarrow} x$. Show from first principles that x_n is eventually equal to x (explicitly $\exists N \in \mathbb{N}$ such that $x_n = x$ whenever $n \geq N$).

3. (6 points) Explicitly construct a sequence (x_n) of real numbers such that $|x_n| \leq 1$ for all $n \in \mathbb{N}$ and $|x_n - x_{n+1}| \underset{n\to\infty}{\longrightarrow} 0$, but (x_n) does not converge. Justify your answer from first principles.

4. (6 points) Given that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is not uniformly continuous, show from first principles that there exist two sequences (x_n) and (t_n) such that $x_n - t_n \underset{n\to\infty}{\longrightarrow} 0$, but the statement $f(x_n) - f(t_n) \underset{n\to\infty}{\longrightarrow} 0$ is false.

5. (10 points) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be defined by $f(x) = \min(3x^2, 1 - x^3)$.

(i) (2 points) Show that f is continuous on $[0, 1]$.

(ii) (2 points) Basing your answer only on (i), explain why f must attain its supremum.

(iii) (6 points) Show that f attains its supremum at a unique point a which satisfies the equation $a^3 + 3a^2 - 1 = 0$.

6. (10 points)

(i) (2 points) Define the concept *Cauchy sequence*.

(ii) (2 points) Define the concept *contractive sequence*.

(iii) (4 points) Let $x_1 > 0$ and $x_{n+1} = 3 + \dfrac{2}{x_n}$ for $n \in \mathbb{N}$. Show that (x_n) is a contractive sequence.

(iv) (1 point) Deduce that (x_n) converges.

(v) (1 point) Find the limit.

7. (10 points) Let $f : [a, b] \longrightarrow \mathbb{R}$ and suppose that f is differentiable at $c \in]a, b[$.

(i) (8 points) Given $\epsilon > 0$ show that there exists $\delta > 0$ such that $|f(v) - f(u) - f'(c)(v-u)| \leq \epsilon(v-u)$ whenever $c - \delta < u < c < v < c + \delta$. Note: Be sure to explain where in your proof you use the fact $u < c < v$ (see (ii) below).

(ii) (2 points) By considering the example $a = -1$, $b = 1$, $c = 0$, $f(x) = x^2$ show that the statement

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is false in general.

8. (10 points) A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to have a slant asymptote $ax + b$ at ∞ if $\lim_{x \to \infty} \Big(f(x) - (ax + b) \Big) = 0$.

(i) (6 points) In this case, if also f is everywhere differentiable and $\lim_{x \to \infty} f'(x) = c$, show that $a = c$.

(ii) (4 points) Find the slant asymptote at ∞ of the function

$$f(x) = \sqrt{x^2 + 2x + 2} + \sqrt{x^2 + 3x + 3}.$$

9. (10 points)

(i) (2 points) State the Mean Value Theorem.

(ii) (2 points) State Darboux's theorem on differentiable functions.

(iii) (6 points) If f is a differentiable function on $\mathbb{R}$ such that $f'(x) \neq 0$ for all $x \in \mathbb{R}$, show that f is monotone. Note: You are not allowed to assume that f' is continuous.

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