FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS MATH 242

Analysis 1

Examiner: Professor S. W. Drury — Date: Friday, 5 December 2008

Associate Examiner: Professor K. N. GowriSankaran — Time: 9: 00 am. – 12: 00 noon.

INSTRUCTIONS

Answer all questions in the booklets provided.
This is a closed book examination.
Calculators are not permitted.

In questions 1-4 inclusive all proofs are to be given from first principles. Only the most basic properties of real numbers may be assumed. Proofs must be self-contained and you must work directly from the definitions of the concepts involved.

This exam has 9 questions and 4 pages

1. (6 points) If $x_n \in \mathbb{R}$, $x_n \underset{n\to\infty}{\longrightarrow} a$ and $x_n \underset{n\to\infty}{\longrightarrow} b$, then show from first principles that $a = b$.

2. (6 points) Prove the following from first principles.

(i) (2 points) $(1+x)^n \geq 1 + nx + \dfrac{n(n-1)}{2}x^2$ for $x \geq 0$ and $n \in \mathbb{N}$.

(ii) (1 point) $(1+x)^{2n+2} \geq \dfrac{n^4x^4}{4}$ for $x \geq 0$ and $n \in \mathbb{N}$.

(iii) (3 points) $n^3 4^{-n} \underset{n\to\infty}{\longrightarrow} 0$.

3. (6 points) Prove from first principles that every increasing sequence of real numbers is either convergent or properly divergent to ∞.

4. (6 points) If $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ and $f'(0)$ and $g'(0)$ exist and if $h : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $h(x) = f(x)g(x)$, show from first principles that $h'(0)$ exists and equals $f(0)g'(0) + f'(0)g(0)$.

5. (10 points) A continuous function $f : [0, \infty[\longrightarrow \mathbb{R}$ satisfies the inequality

$$|f(a+1) - f(b+1)| \leq |f(a) - f(b)|$$

for all $a, b \in [0, \infty[$. Show that f is uniformly continuous on $[0, \infty[$. Any theorem or theorems that you use in your proof should be carefully stated.

6. (10 points) Let $a > 1$ and let $f(t) = a\dfrac{1+t}{a+t}$.

(i) (1 point) Show that $|f'(t)| \leq \dfrac{a-1}{a}$ for $t > 0$.

Let $x_1 > 0$ and suppose that x_n is defined inductively by $x_n = f(x_{n-1})$ for $n = 2, 3, \ldots$

(ii) (1 point) Show that $x_n > 0$ for all $n \in \mathbb{N}$.

(iii) (2 points) Define the concept *Cauchy sequence*.

(iv) (4 points) Show that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.

(v) (2 points) Deduce that $(x_n)_{n=1}^{\infty}$ is convergent and identify the limiting value.

7. (10 points)

(i) (3 points) State the Bolzano–Weierstrass Theorem.

(ii) (7 points) Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers with the property that every subsequence of $(x_n)_{n=1}^{\infty}$ which converges to a finite limit converges to 0. Show that

$$\frac{x_n}{x_n^2+1} \to 0 \text{ as } n \to \infty.$$

8. (10 points)

(i) (2 points) State Taylor's Theorem with the Lagrange form of the remainder.

(ii) (6 points) Let $m \geq 0$ and $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function with first and second derivatives existing everywhere on $]a,b[$ and $f''(x) \geq -m$ for all $x \in]a,b[$. Show that $f(x) \leq \max(f(a), f(b)) + \frac{1}{8}m(b-a)^2$ for all $x \in [a,b]$.

(iii) (2 points) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function with first and second derivatives existing everywhere on $]a,b[$ and $f''(x) \geq 0$ for all $x \in]a,b[$. Show that $f(x) \leq \max(f(a), f(b))$ for all $x \in [a,b]$.

9. (10 points) Let $a < c < b$. Let $f :]a, b[\longrightarrow \mathbb{R}$ be continuous and suppose that $f'(x)$ exists for all $x \in]a, b[\setminus \{c\}$. Suppose that $\lim_{x \to c} f'(x)$ exists and equals γ. Show that $f'(c)$ exists and equals γ. State carefully any theorem that you are using in your proof.

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